Report on the TI-Nspire experimentation in two Italian classrooms 2007-2008

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Introduction

This document is divided in three main parts and two appendices; it contains the report on two teaching experiments developed from 2007 to 2008 in two Italian classes, one in the first two years of secondary school (grades 9-10), the other in the last two years of secondary school (grades 12-13).

Part I contains the report on the experiment teaching made in the class of Domingo Paola in Finale Ligure (Liceo Issel: grades 9-10).

Part II contains the report on the experiment teaching made in the class of Pierangela Accomazzo in Torino (Liceo Einstein: grades 12-13).

Part III contains an analysis of the experimentation results.

Appendix A, in a separate file, consists in the lecture notes given to prospective teachers of mathematics by F. Arzarello in two courses (in English) about statistics approached with TI-Nspire: I am grateful to C. Andrà and Duong Pham Trieu for having collected and edited this material, which can be useful also for in-service teacher training.

Appendix B contains the scientific part of the proposal, as it was presented to Texas in October 2006.

Two folders are attached to the report (Accomazzo and Paola), where one can find all the TI-Ns problems given during the teaching experiment in the two classes and some other documents referred to in the report.

I thank Texas Instrument for giving me the opportunity of an exiting research and all the people involved in it: the two experimentation teachers, Pierangela Accomazzo and Domingo Paola for their enthusiasm; my colleage, Ornella Robutti, for her inspiring suggestions; my collaborators, Francesca Ferrara, Cristina Sabena, Chiara Andrà, Duong Pham Trieu, for their help; and the students Lorella Allais and Silvia Damiano, for their long work during the two years of experimentation.

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PART I
The classroom of D. Paola

Approaching Functions with TI-nspire (I and II year of Liceo Scientifico; 14-16 y.o.)

Domingo Paola
Liceo scientifico 'Issel' – Finale Ligure
Corso PNI

In the first year of the teaching experiement the class (9th grade) had 22 students, 10 girls and 12 males. The class was involved in a PNI experiment (national computer science project for upper secondary schools). Each week the class had 5 slots for mathematics and three slots for physics, with different teachers: each slot lasts 50 minutes. The school is the Liceo Issel (Finale Ligure). The same teachers follow the same class for the five years of the secondary schools (from grade 9 to grade 13).
In the first experiment year (2006-2007) the students have been available to engage in the TI-Nspire testing in the classroom. But their availabilty for the whole classroom discussions has not been very high; moreover anly some students have been available to work at home in a serious and systematics way. The final results have not been so brilliant: only three students have got excellent marks at the end of the year, while six of them have had a negative mark and two of these have not been admitted to the next year. Moreover two girls with negative mark have changed the class.
In the second experiment year only 18 of the 22 students of the first year were still present (7 girls and 11 males); there were two more students (a male and a female) who had to repeat the year. Their engagement has increased, even only 40% of them worked also at home on TI-Nspire. At the end of the second year no student had negative marks for mathematics, even if three of them had to work during summertime to consolidate their fragile preparation in this topic.

The description of the work in the two years.
Goals (designed and achieved) in the first year[1].

a) First steps in grasping the concept of function as a variable quantity with respect to another (at first stance as a quantity that changes in time);
b) Building the concept of linear function and acquiring the main notions in linear function algebra;
c) Knowing the discrete linear systems and the exponential growth;
d) First steps in linear regression concept.

Goals (designed and achieved) in the second year:

a) Building the concept of quadratic successions and functions;
b) Knowing the square root function;
c) How to linearly approximate a function in one of its points;
d) Hints about the relationships between a function and its (possible) derivative and its primitives;
e) Models of linear regression.

[1] About 80% of the hours in the year have been dedicated to these goals; the remaining 20% have been used for achieving some competencies in algebraic techniques.

Learning design in the first year[2].

<u>First phase</u>: September-October

Sensory-motor approach to the concept of function (using motion probes). The main goal is to link the concept of function as a quantity that varies in time and its variation modalities (does it increase-decrease and how?) to sensory-motor experiences. The idea is that such an approach can promote suitable conditions for building the function in a robust and permanent way, based on its cognitive roots, so that it can be used positively in later activities, when the symbolic and formal aspects become more important.

<u>Second phase</u>: November-half of March

1. (First and second) finite differences for numerically approaching the concepts of "constant increasing/decreasing", "it increases/decreases more and more", "it increases/decreases less and less": In particular, it is considered the case when the independent variable does not change with a constant step.
2. The concept of slope from a numerical, graphical and formal-symbolic point of view. Linear functions: graphical, numerical and symbolic-formal approach. Zeros and sign of a linear function. The algebra of linear functions (particularly with the proof the composing two linear functions one still gets a linear function).
3. Quadratic and polynomial growths. Exponential growths.

<u>Third phase</u>: half March-May

1. Discrete linear dynamical systems from a graphical, numerical and formal point of view.
2. Linear regression.

Methodology

The students have worked from November to May about 3 hours over 5 with TI-Nspire (in the first year in the school laboratory without handheld; in the second year in the classroom using handhelds) solving problems posed by the teacher. The students worked in pairs or in small 3-4 persons groups; moreover, there have been systematic mathematical discussions of the whole classroom, orchestrated by the teacher in order to systemize and organize students knowledge and observations.

For each activity we have done systematic observations. Some of the activities have been videotaped. In these last activities we have asked the students:

a) to think individually to the posed problem for 5 minutes, without using any tool (neither pencil and paper);

b) to think individually to the posed problem for 5 minutes possibly using pencil and paper

c) to discuss for 10-15 minutes the possible solution(s) with their mates;

d) to solve the problem using also TI-Nspire;

e) to write a document in TI-Nspire that described their solution processes and their final products.

Below we describe:

a) the didactical path that has been followed to teach the concepts of polynomial and exponential growth, and particularly an activity at the end of the first year using the PC's in the laboratory of the school, when the students had completed the first two points of the second phase listed above.
b) the didactical path for the quadratic functions (point c of the second year)
c) the didactical path for the first elements of calculus (point d of the second year)

[2] All the didactical materials are downloadable at: http://www.matematica.it/paola/Corso%20di%20matematica.htm .

a) **The didactical path for the polynomial and exponential growth**

All the activities of discovery, of concept building, problem solving are done in pairs of students who are balanced for their results in mathematics; all the activities of recovery, strengthening, reviewing are done by pairs with an unbalanced preparation. By didactic contract, every hour of work in the school must by joined by half an hour work at home.

Below, in the activities with TI-Nspire, we list:

a) a link to a TI-Nspire file (the files are in Italian: they are in the folder Domingo);
b) a short description of the activity (boldface);
c) (some parts of) the activity itself (in blue).

MOD01.tns

Activity for systematizing the fact that the first differences are constant if and only if the growing is linear. In other words the linear changes of y with respect to x are featured as those whose first differences are constant and proportional to the slope of the linear function $y = f(x)$. The activity is completely guided and exposes the students to the numerical, graphic and symbolic approaches. The text of the activity starts as follows:

We have seen that a quantity y changes linearly with respect to another quantity x iff the first differences of y (represented by Δy) are constant, that is they do not change with x. This fact can be represented from different points of view: numerical (as suggested by the spreadsheet), graphical (see the straight line drawn in the Cartesian plane within Graph and Geometry in this page), symbolic (see the space in this page for the calculator application: we will come back later to the formulas in it).

MOD2.tns

The activity gradually accompanies the students to understand that what they have seen when the first differences are constant (for linear growths) can be extended by analogy to the constant second differences and this features the quadratic growths. The text of the activity is the following:

We have seen that, given a table with some coordinates of a linear function, it is possible to get the formula of the linear function that has such values. We have seen that the first differences are constant iff the function which has the values in the table is constant. Then we have explored the relationships that link the slope of a linear function to its first differences.

Now we wish to go on in this work about functions: we ask ourselves if it is possible to get information on the features of functions even when the first differences are not constant. In this activity we will explore what happens when the first differences vary linearly. In mathematics often one makes analogies with situations that have already been considered. The analogy may suggest ideas whose validity is not sure at all but must be checked carefully. Sometimes the analogy is adequate for solving the new problems, but sometimes it is not so.

In our case (functions whose first differences vary linearly), using analogies we could conjecture that, since with constant first differences we get linear functions, with first differences that vary linearly we get quadratic functions.

How can we explore if the conjecture makes sense? A first idea could be to consider quadratic functions, namely of the type f(x) = a*x^2 + b*x + c and to check their first differences; to say that first differences vary linearly means that their first differences, that is the second differences of the function, are constant. Hence build a spreadsheet with the x column where values change with a unitary step, those with values of the y got applying to the x values a quadratic formula (the x must appear with a square power), for example with the formula 3*a2^2 + 4*a2 - 5, a column with the first differences and a column with second differences.

Observe what happens; then modify the formula and check each time what happens to first and second differences. What can you conclude?

During this activity some pairs of students already build a spreadsheet like that suggested in the following activity 3.
A couple of students substitutes the absolute references to study what happens varying the coefficients a, b, c in $ax^2 + bx + c$, with the letters in the cells. These students spontaneously use the spreadsheet as a symbolic tool.

MOD3.tns
The students who have not yet done it are guided to build a spreadsheet that, suitably using the absolute and relative references, allow them to explore how a growing is quadratic iff the second differences are constant. The text of the activity starts as follows:
In the previous activity we have asked you to make some explorations to observe the changes of the first differences of quadratic function. Remember that a quadratic function can be represented by the following expression: f(x) = a*x^2 + b*x + c.
In this activity:

a) We give you a spreadsheet (page 2) already prepared and good for our aims. We ask you to compare with the one you have already built in the previous activity, pointing out possible limits and potentialities with respect to the one you used previously (write here your observations);
b) We propose a justification of the observed fact namely that "if the first differences of a function vary linearly, then the function is of the second degree".

The sheet ends asking to do the following activity:
To do at home (singularly or in group)
Claim: "an example of quadratic succession is that of the areas of the squares whose side increases at each step of a fixed quantity starting from a given one". Justify with examples and observations the claim, using TI-Nspire.
Propose at least another example of a quadratic function.

MOD04.tns
Activity for the students, according to the following task:
Even if we do not do it now, it is possible to prove that a polynomial function is of degree *n* iff its first differences are of degree *n*-1, or iff its *n*-th differences are constant. Check that a 3th degree function has the first differences that vary quadratically, the second differences that vary linearly and the third differences that are constant. Use TI-Nspire: inspire yourselves from the previous activities.
A typical third degree succession is given by the volumes of cubes whose side increases at each step of a fixed quantity starting from a given one. Justify this claim through examples and observations and then give at least one more example of a third degree succession.

MOD05.tns
Activity that introduce to exponential growths.
In previous activities you have seen that in case a function is polynomial, and conversely, then the repeated passage to the finite differences as many times as the degree of the polynomial allows to get a constant value. To say it in another way, the "polynomial variations" are characterised by the possibility of getting a differences succession that is constant. In case the polynomial is of degree n, the constant succession is the n-th (that one gets applying the operation of passage to the differences n times). For example, if the function is of fourth degree, the fourth differences are constant; if it is of fifth degree, its fifth differences are constant, end so on.
Now we will study variations that are different from those of the polynomial, but are very important for modelling real phenomena: the exponential variation. They are characterised by a constant rate between two successive values.

For an example, consider the following spreadsheet, where in the y column you find the values of a quantity that vary exponentially. In fact you can observe that the value in a cell (from the second on) is got multiplying by 2 the previous one. Hence the ratio between two consecutive cells is constant. In the third column you find the ratios between two consecutive values; we have written also the differences up to the fourth differences. As you can see, they are not constant. If you wish you can compute also the fifth, sixth, seventh, ...differences: you never get constant values. On the contrary, can you realise a property that features all the differences? Check them carefully and then report here the results of your observations.
N.B. From now on, for tables where the variable x changes with a constant step equal to 1, we will speak of "sequences of values" or even of "successions" (imagining to continue the sequence of values *ad infinitum*)

The work has gone on with other activities about polynomial and exponential growths and with some activities linked to discrete dynamic models. Here we do not report the files, since they are applications of the activities above. Instead we will report with some details the activity about the transition to the quadratic growths: see MOD02.tns above for the task.

The students, when this activity was proposed had reached a good level of instrumentalisation (see part III) of the TI-Nspire spreadsheet. For this reason they have not met any difficulty in checking for the proposed example that the second differences are constant; various pairs of students have been able autonomously to organise a TI-Nspire spreadsheet that could support their exploration and to do that they used the technique of the absolute references (*fig.1*).

	A	B	C	D	E	F	G	H	I	J	K
◆											
1	x	f(x)	d(f(x))	dd(f(...	x0	a	b	c	h		
2	0	-5	7	6	0	3	4	-5	1		
3	1	2	13	6							
4	2	15	19	6							
5	3	34	25	6							
6	4	59	31	6							
7	5	90	37	6							
8	6	127	43	6							
9	7	170	49	6							
10	8	219	55	6							
11	9	274	61	6							
12	10	335	67	6							
13	11	402	73	6							
14	12	475	79	6							
15	13	554	85	6							
16	14	639	91	6							
17	15	730	97	6							

B2 | =\$f\$2·a2^2+\$g\$2·a2+\$h\$2

fig. 1

The formulae that generate the numbers in the spreadsheet are:

a) in the cell A2: = \$E\$2, so that in this cell you find the first value of the independent variable, which in fact has been inserted in E2 cell (the formula refers to its absolute reference, because of the character \$);

b) in the cell A3: =A2 + I2, so that the initial value of the independent variable, which is in A2 cell, is repeatedly increased with the value present in I2 cell (in the case of the figure the increment of the independent variable is 1);
c) in the cell B2: = F2*A2^2 + G2*A2+H2, so that one gets the value of the quadratic function when the independent variable assumes the value in the corresponding cell in column A. The cells to which refer the absolute references contain the values of the coefficients of the quadratic function;
d) in the cell C2: = B3 – B2, so that one gets the values of the first differences;
e) in the cell D2: = C3 – C2, so that one gets the values of the second differences.

Modifying the values in the cells E2, F2, G2, H2, I2 one produces changes in the initial value of the independent variable; in the first, second and third coefficient of the quadratic function; and in the step of the independent variable. Hence it is very easy exploring what happens changing one of these values: all the spreadsheet numbers change consequently, so that the students can observe what does changes and what does not. For example, the students realize at once that if one changes only the third coefficient, only the B column changes (that of the function values); in particular the values of first and second differences remain the same (*fig.2*).

	A	B	C	D	E	F	G	H	I
◆									
1	x	f(x)	df(x)	ddf(x)	x0	a	b	c	h
2	0	7	-1	-4	0	-2	1	7	1
3	1	6	-5	-4					
4	2	1	-9	-4					
5	3	-8	-13	-4					
6	4	-21	-17	-4					
7	5	-38	-21	-4					
8	6	-59	-25	-4					
9	7	-84	-29	-4					
10	8	-113	-33	-4					
11	9	-146	-37	-4					
12	10	-183	-41	-4					
13	11	-224	-45	-4					
14	12	-269	-49	-4					
15	13	-318	-53	-4					
16	14	-371	-57						
17	15	-428							

fig. 2

The students, who know how to use first and second differences to check if a succession increases or decreases and also how it increases or decreases, conjecture that the growth and the concavity of a quadratic function do not depend on the coefficient c in the formula $ax^2 + bx + c$.

Modifying the coefficient b produces effects both on the values of the function and on the column of first differences, while that of the second differences does not change (*fig.3*).

	A	B	C	D	E	F	G	H	I
◆									
1	x	f(x)	df(x)	ddf(x)	x0	a	b	c	h
2	0	3	2	-4	0	-2	4	3	1
3	1	5	-2	-4					
4	2	3	-6	-4					
5	3	-3	-10	-4					
6	4	-13	-14	-4					
7	5	-27	-18	-4					
8	6	-45	-22	-4					
9	7	-67	-26	-4					
10	8	-93	-30	-4					
11	9	-123	-34	-4					
12	10	-157	-38	-4					
13	11	-195	-42	-4					
14	12	-237	-46	-4					
15	13	-283	-50	-4					
16	14	-333	-54						
17	15	-387							

fig. 3

Various pairs of students autonomously make the conjecture that the coefficient *b* determines the growing of a quadratic function but not its concavity; while changes of *a* produce effects not only on the values of the function and of its first differences but also on the column of second differences: hence it is the first coefficient to determine the concavity of the quadratic function. Modifying the values in E2 cell (the first value of the independent variable) produces changes in all the columns, except in that of the second differences, while a change a change of the variable increment (I2 cell) generally modifies also the column of second differences. From this observation one may conjecture that there is a relationship between the first coefficient *a*, the second differences and the increment step of the independent variable. This conjecture is less transparent than the previous ones, but if suggested by the teacher, it is grasped by the students. The problem consists in determining what is this relationship. Probably, leaving a lot of time to the students for exploring maybe they can determine it by trials. But the gain in terms of understanding with respect to the time is not worthwhile. So we have not left so much time to the students. We have decided to suggest the students to use the power of the symbolic characters of the TI-Nspire spreadsheet. An interesting aspect of this activity is that a couple of students has anticipated the teacher's suggestion and has tried to substitute letters to numbers in the cells E2, F2, G2, H2, I2: the result is represented in the figure below (*fig.4*).

	A	B	C	D	E	F	G	H	I
◆									
1	x	f(x)	df(x)	ddf(x)	x0	a	b	c	h
2	x0	a*x0^2+b*x0...	a*(h^2+2*h*x0)+...	2*a*h^2	x0	a	b	c	h
3	h+x0	a*(h+x0)^2+...	a*(3*h^2+2*h*x0...	2*a*h^2					
4	2*h+x0	a*(2*h+x0)^2...	a*(5*h^2+2*h*x0...	2*a*h^2					
5	3*h+x0	a*(3*h+x0)^2...	a*(7*h^2+2*h*x0...	2*a*h^2					
6	4*h+x0	a*(4*h+x0)^2...	a*(9*h^2+2*h*x0...	2*a*h^2					
7	5*h+x0	a*(5*h+x0)^2...	a*(11*h^2+2*h*x...	2*a*h^2					
8	6*h+x0	a*(6*h+x0)^2...	a*(13*h^2+2*h*x...	2*a*h^2					
9	7*h+x0	a*(7*h+x0)^2...	a*(15*h^2+2*h*x...	2*a*h^2					
10	8*h+x0	a*(8*h+x0)^2...	a*(17*h^2+2*h*x...	2*a*h^2					
11	9*h+x0	a*(9*h+x0)^2...	a*(19*h^2+2*h*x...	2*a*h^2					
12	10*h...	a*(10*h+x0)^...	a*(21*h^2+2*h*x...	2*a*h^2					
13	11*h...	a*(11*h+x0)^...	a*(23*h^2+2*h*x...	2*a*h^2					
14	12*h...	a*(12*h+x0)^...	a*(25*h^2+2*h*x...	2*a*h^2					
15	13*h...	a*(13*h+x0)^...	a*(27*h^2+2*h*x...	2*a*h^2					
16	14*h...	a*(14*h+x0)^...	a*(29*h^2+2*h*x...						
17	15*h...	a*(15*h+x0)^...							

fig. 4

The relationship among the second differences, the coefficient a and the step h is clear; it is written in the spreadsheet: the second differences equal $2ah^2$. Using symbols in this case does not produce a loss of meaning, as usually may happen in the transition from the numerical to the algebraic register: on the contrary, the formula produced by the spreadsheet condenses the meaning so got after a long time spent in making experimentations within the numerical register. In this case the symbols are a sort of lens that allows seeing in a transparent way what is hidden under the complex variability of numbers. This is an important point, about which the students are asked to think. This issue has been accomplished in the following systematizing lecture. In fact a meaningful approach to symbols in mathematics must by joined to reflections about them and their functions in the mathematical discourse and in problem solving. Of course we cannot say to have proved the claim that in a second degree function the second differences are constant and equal to $2ah^2$. The spreadsheet shows only that this relationship holds for the data in its cells (that are in a finite number). To prove it in general we need more sophisticated tools, which are beyond the goals of the first years of secondary school.

b) The path on quadratic functions (second year)

It has the following phases:

1. We recall the method of finite differences from the first year (with some home works and explanations in the classroom, when necessary)
2. From quadratic growths to quadratic functions;
3. Second degree problems.

In each moment of the activities we have underlined the graphical, the numerical and the symbolic aspects. We have also tried to make the students reflect on the potentialities and the functions of some actions made possible by TI-Nspire: in particular we have dedicated some time to discuss the tools “calculator”, “variable declaration”, “automatic data capture”, “symbolic spreadsheet”.
Globally this path has required two months in the classroom (assessment activities included).

Classroom Activity 1 (1 hour in the laboratory)

The first activity has used the software Graphic Calculus, by David Tall: according to the author this software should be a generic organizer for learning Calculus concept. The students had already used it when studying linear functions.

The module "parabola" of GC introduces a screen like that in *fig.5*. As one can see, the graphical region is the most important; on the right one has the formula and the zeroes (if any) of the function. The students can act on the red and green points, modifying the position of the parabola (acting on the vertex they can translate the parabola; if they act on another point, e.g. the red one of the figure, they can modify the opening and the concavity of the parabola). The task is to translate the vertex and to observe what relationship does exist between the motions and changes of the parabola formula.

Generally Graphic Calculus gives two formulas for the parabola (see *fig. 6*); the students must also be able to prove the equivalence and to understand what of the two formulas allows them to do better the task they have been given.

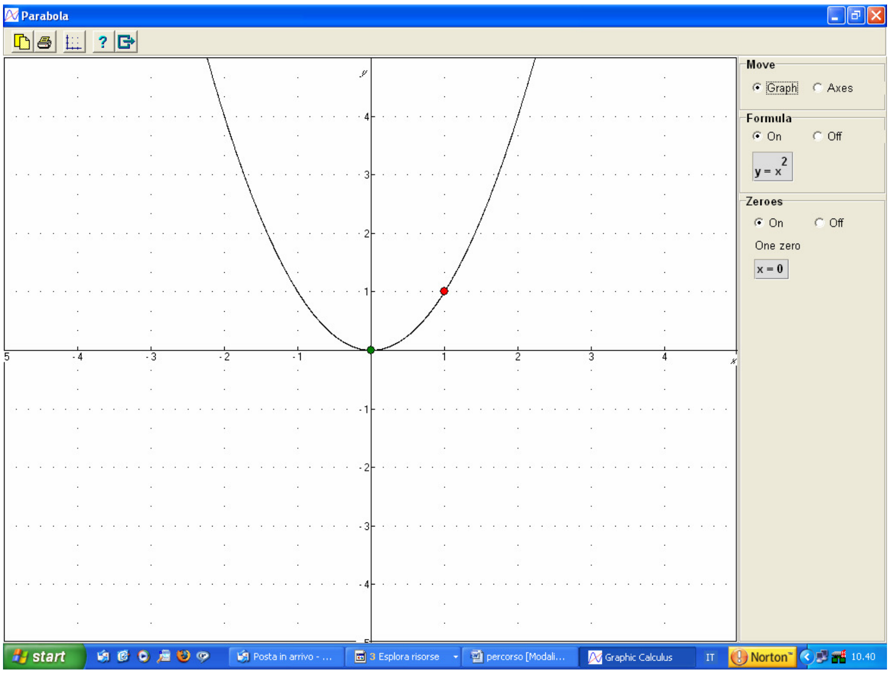


fig. 5

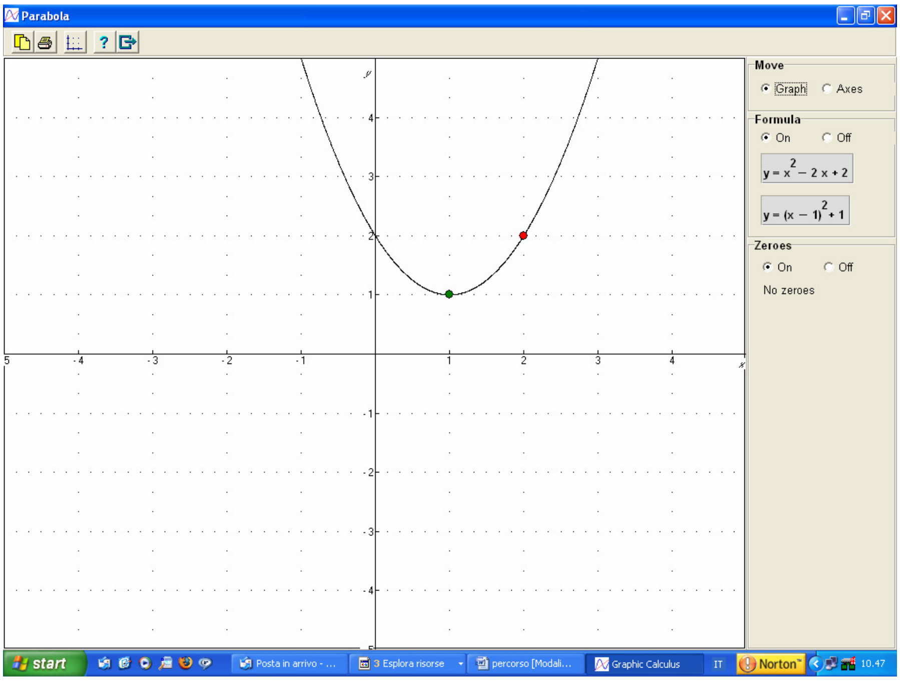
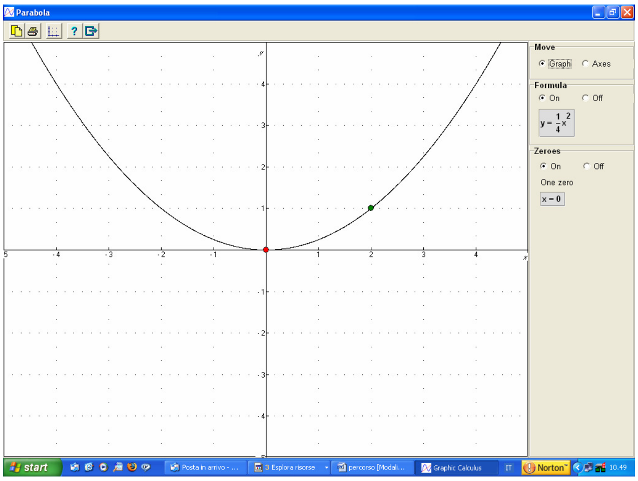


fig. 6

Later the students must explore what happens to the formula when acting on the openness of the parabola (*figs. 7, 8*).



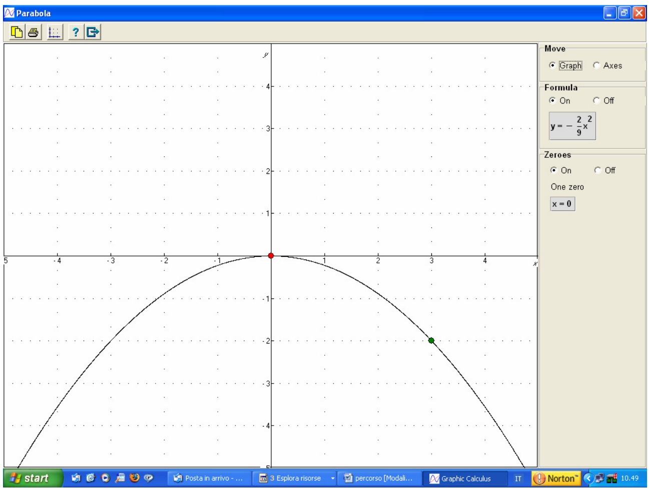


fig. 7

fig. 8

Before the first activity and then between the first and second one, the students have also worked to some problems that they had already appproached the year before (i rettangoli isoperimetrici, il principio di Fermat, and the area of isosceles triangles where the length of two equal sides is given) with the following goals:

a) To become acquainted with the handhelds and the new TI-Nspire version;
b) To reflect on the potentialities of the more common instrumental actions (“compute”; “variable declaration”; “automatic capture of dates”; “scatterplot graph”);
c) To recall the techniques of the first and second differences;
d) To reflect further about the meaning behind the choice of the independent variable.

Activity 2 (2 hours of lecture, exercises included)
With the help of the teacher the students systematize the observations made in the computer science laboratory. At the end the students must acknowledge the formula $f(x) = a(x - x_v)^2 + y_v$ as the expression of a generic function of second degree with the vertex in the point $(x_v ; y_v)$ and the openness ruled by the coefficient a.
Moreover, using this formula, they must determine the zeroes and the sign of a quadratic function even they do not yet know the formula for the second degree equations. Some exercises have been developed at the blackboard with the participation of all the students.

Activity 3 (1 hour of lecture, exercises included)
With the help of the teacher the students learn how to transform a generic expression of the type $ax^2 + bx + c$ into the form $a(x - b/(2a))^2 + (-b^2 + 4ac)/(4a)$. In such a way they identify the coordinates of the vertex of a parabola in function with the coefficients of a quadratic function written as $ax^2 + bx + c$.

Activity 4 (2 hours in the computer science laboratory) Motion of a projectile
The students are asked to simulate the motion of a projectile using TI-Nspire. At the beginning the students must fix the modulus of the velocity and build a simulation that shows how the trajectory changes in function of the fire angle with respect to the horizontal line. Successively they must study how the range changes with the fire angle. The students have also approached the problem with their physics teacher, who has introduced the motion equations on the vertical and horizontal component (neglecting all the forces, except the weight).

We waited that the students were able to build a file similar to the following one: moto di un proiettile.tns (the explanantion how the file has been built is at the end of this activity). The students have built such TI-Nspire files using a learning platform, where they could share their works (this practice is usually followed for all mathematical and other disciplines activities). The teacher had previously explained the students how in graphic representations of TI-Nspire it is possible to represent a curve parametrically.

The main difficulties met at the beginning by the students consisted in identifying the problem variables and parameters. In particular they confused variables and parameters: for example they have considered x, y and the angle as "the same". Moreover they had difficulties in distinguishing between the time (the independent variable with respect from which the two coordinates of the projectile positions depend) and the angle (variable from which the range depends). Some students have solved the problem almost autonomously, interacting with the teacher, who has not given real answers but has posed questions like: "What difference between the coordinates of the projectile and the time?", "What does the angle represent? Does it change with x and y? Does it change with time as x and y?". For other students it has been necessary to clarify that each time a projectile is fired the angle and the modulus of the initial velocity are fixed (parameters of the problem). Once that the angle and the initial speed have been fixed the trajectory must be studied (namely the way the ordinate of the projectile centre of mass changes with respect to the its abscissa). Successively they had to study how the trajectory changes in function of the angle, provided the modulus of the velocity does not change. In the end they had to see how the range depends from the angle. Five pairs needed detailed explanations. For example, two students who had taken **vx** and **vy** as variables (they were the projections on the two axes of a vector OP that follows a point P on a circle of center O: see below), used three columns in the spreadsheet to represent the time (column A, with constant step = 1), formula =**vx***A1 (column B), the formula =**vy***A1-4.9*A1^2 (column C). The laws are correct, but the two variables $\mathbf{v_x}$ and $\mathbf{v_y}$ as, because of their definition, varied with the fire angle and the students acquired their values with the variation of that angle. So they used in the two formulas instead of the (constant) projections of the initial velocity, two variables that depended on the fire angle (which was varied by the students themselves). Of course their graphic had no meaning with respect to the sought trajectory.

Provided such explanations, the files have been structured by the students as follows:

a) They have drawn a circle with centre in O and with a fixed radius (it corresponds to the modulus of the initial velocity);
b) They have taken a point P on the circle;
c) They have considered the vector OP;
d) They have considered the angle AOP, where A is the intersection point of the circle with the positive x axis;
e) They have measured the angle AOP;
f) They have considered the projections H, K of P resp. on x and y axis;
g) They have measured OH and OK;
h) They have declared the variable **vx** (measure of OH), the variable **vy** (measure of OK) and the variable **angle** (measure of the angle);
i) They have defined and drawn the graph of the function(in parametric form) $x(t)$ = **vx***t and $y(t)$=**vy***t-0.49*t^2, with t between two suitable values (e.g. 0 and 20);
j) Using "solve" (or with paper and pencil) they have got the range in function of **vx** e **vy** (=**vx*vy** / g);
k) Using "text" they have written vx*vy / g and then they calculate it;
l) They define the variable **gittata**;
m) The automatically capture **angolo** and **gittata**;
n) They study the function gittata=f(**angolo**) using the finite differences and the scatterplot graphic.

Acivity 5 (1 hour in the laboratory) Finding the formula of Graphic Calculus

The students, in inhomogeneous pairs, use the module "finding the formula" of Graphic Calculus, which gives graphs of second degree functions, of which the students must determine the formula. Moreover they work on the use of the formula $y = a(x - x_v)^2 + y_v$ (when the vertex is detachable on the graph) and on the condition under which a point belongs to a curve (to determine a, when the vertex is detachable, otherwise to determine a, b, c).

Activity 6 (1 hour in the classroom: systematizing the work on the projectile motion and on "finding the formula")

It is a systematizing lecture.

Activity 7 (1 hour: individual test on quadratic functions)

Here is the text of the test:

1. Draw the following quadratic functions and successively determine their zeroes and signs. Explain your procedure.

$f(x) = 2(x^2 - 1) + 3 \qquad g(x) = 3x^2 - 1 \qquad h(x) = -4x^2 + x - 1$

2. Determine the formula for the quadratic function with vertex in (2,3) and passing through A(1, 1).

Activity 8 (2 hours in the classroom: individual test with TI-Nspire handheld)
Here is the text of the test:
Consider a rectangle ABCD, whose sides AB and BC measure resp., a, b (a > b). Take a variable point M on BC and a variable point N on AB, such that AN = BM. What can you say about the perimeter and the area of the triangle NBM? Justify your answers.
The students used the handheld but at the end they have written their answer on paper sheets.

Activity 9 (1 hour with view screen and TI-Nspire handheld)
Two pairs have presented the classroom and the physics teacher the result of their work on the projectile motion (some files have been inserted in the virtual classroom).

Activity 10 (1 hour)
Comment to the final test.

Activity 11 (1 hour)
Systematization about the work on the zero research and the sign determination of quadratic functions and on the determination of the quadratic functions that pass through three points.

Attività 12 (1 hour, individual testing on quadratic functions)
Here is the text of the test:
1. Draw the following quadratic functions and successively determine their zeroes and signs. Explain your procedure.

$f(x) = -5(x^2 + 1) + 3$ $\quad g(x) = 3x^2 + 1$ $\quad h(x) = -2x^2 + x - 1$

2. Determine the formula for the quadratic function passing through A (1; 1), B(2; 2) and C (3; 7).

Activity 13 (2 hours working in pairs in the laboratory: the two squares problem)
Here is the task (videorecorded):
Given a segment AB of a fixed length, consider a point C on AB and build on AC and CB, from the same side with respect to the line AB, two squares of sides Ac and CB, resp.. What can you say about the area and the perimeter of the figure made by the two squares? Justify your answers.

Activity 14 (1 hour)
Comment on the video-records of the previous activity.

Attività 15 (2 hours)
Systematizing the previous activities.

Activity 16 (2 hours)
Systematizing the previous activities.

Activity 16 (2 hours in the laboratory)
Simulation of the Graphic Calculus environments in TI-Nspire (the activity has been videorecorded).
In the laboratory the students work in homogeneous pairs; the file parabole1.tns is proposed to them. Here is its task:
In this activity we will learn to build a slider, that is a cursor which allows to vary the values of a parameter moivin a point on a segment. We will use three sliders to change the values of a quadratic function like f(x) = a*x^2 + b*x +c or f(x)=a(x-b)^2+c.
Let us fix a segment AB of length *d* and two numbers of values v1 and v2. Let us consider a point C on AB and let us call x the variable length of AC. Let us consider the expression v1+x/d(v2-v1).
We observe that:

1. It holds v1 if x = 0 (C = A);

2. It holds v2 if x = *d* (C = B);
3. While x varies between 0 and d, it is the result of the addition of v1 with a term that is proportional to x. The constant of proportionality is the ratio among th difference between v2 and v1 and the length d of AC. If v2 – v1 = d, the coefficient is 1. Since v2 and v1 are any numbers, we can interpret the expression v1+x/d(v2-v1) as a variable value that starts with v1 and is incremented of a quantity x/d(v2-v1), which can be made as small as wished increasing d (once v2 and v1 are fixed).

The expression v1+x/d(v2-v1) is a good representation of a cursor.

In this activity we will build three cursors:

a) the cursor *a*, that determines the coefficient *a* of the quadratic function;
b) the cursor *b*, that determines the coefficient *b* of the quadratic function;
c) the cursor *c*, that determines the coefficient *c* of the quadratic function;

Let us fix the extremes of variation v1 and v2 for the three cursors *a, b, c*. We wish that the in all cases they are symmetric with respect to 0. Let us fix the extremes -2, 2 for *a*; -4, 4 for *b* and -8, 8 for *c*.
In the next page the three cursors are realized for a lenth of d = 10.
With "declaring variables" to each of the three cursor a variable (resp. *a, b, c*) has been assigned.
In the third page a function f1(x) = a(x-b)^2+c is given and in the fourth page a function f2(x)=a*x^2+b*x+c: their coefficients vary from the cursors, hance varying the point C on AB. A problem is that varying C, all a, b, c vary. How can you modify this worksheet so that a, b, c can vary the one independently from the other? Build up such a worksheet.
You are asked to produce two pages, divided into two parts:

- in the first one, in the one part it must be possible to vary the coefficients independently and in the second part one must see the effects of such variations on f1(x) = *a*(x-*b*)^2+*c*;
- in the second one, in the one part it must be possible to vary the coefficients independently and in the second part one must see the effects of such variations on f2(x)=*a**x^2+b*x+*c*.

What is the meaning of *a, b, c* in the two cases? Did you expect what you see graphically varying the coefficients? In particular, what is the geometric meaning of *a* in the first and in the second case? And of *c*? and of *b*? What happens in the second case if a and c are fixed and b varies? Justify your answers.

NB. The task is taken from: P. Drijvers, "Learning Mathematics in a computer algebra environment: obstacles are opportunities", ZDM, vol. 34, §5 (2002).

The students have 20-30 minutes to explore and, interacting with the teacher, they arrive to pose the following question: what geometric locus do the vertexes of the parabolas $y = ax^2 + bx + c$ describe, while b varies and a, c are kept fix?

At the end in the systematizing lecture, commenting the different strategies proposed by the students, the teacher has synthesized them in the following one:

a) Detect the vertex of the parabola $y = ax^2 + bx + c$ taking a point P on it, drawing a straight line r through P parallel to the x axis, considering the further intersection T between the parabola and r, building TP, namely the parabola axis and intersecting such an axis with the parabola;
b) Detect the coordinates of the vertex and define them as variables **xv, yv**;
c) In a spreadsheet automatically collect the data **b**, **xv** and **yv**;
d) Draw the scatterplot of **xv** and **yv**;
e) Check on the thumb if the curve looks like a parabola;

f) Check using the first and second differences that **xv** varies linearly with b, while **yv** varies quadratically with b;
g) Determine the symbolic relation yv = f(xv) and check that is a parabola of equation $y = -ax^2 + c$.

Everything is in the file parabole3.tns.

Activity 18 (4 hours work in the classroom, using handhelds) Second degree problems.
These activities consolidate the students' knowledge about second degree problems. Here is the text of some of the proposed problems:

1. A square ABCD of side a is given. Let M be on side AB. On the side AM build a square AMNP within ABCD. Join the vertex N with D and C. What can you say of the area of the figure made by the square AMNP and by the triangle NCD? Justify your answers.
2. Tossing an unfair coin the probability that the side T appears is p, while the probability that the other side C appears is 1-p. What is the probability of getting in two successive tosses one T and one C? What must p be so that the probability of getting TC or CT in two successive tosses is 0.7? And what if the probability is not to be bigger than 0.3?
3. From a square with side 1 one cuts off four small isosceles, rectangles triangles with side x, getting an octagon. For which values of x the octagon has an area that is 2/3 of the square area? And for what values of x the area of the octagon is less of 1/3 of that of the square?
4. The function "weekly profit" of a product is given by $P(x) = -0.0001x^2 + 3x - 12\,500$, where x is the number of units produced in a week and P(x) is its profit (in euro). What is the maximum weekly profit? How many units must be produced in one week to get the maximum profit? How many units must be produced in one week to get a profit of at least 70% of the maximum profit?
5. The variation of the height h in function of the time t of an object thrown upside from the earth is given by the formula $h(t) = -5t^2 + 10t$ (time in seconds and height in meters; the resistance of the air is neglected). What is the maximum height reached by the object? After how many seconds the object reach this maximum height? After haw much time the object falls down to the earth? After how much time does it reach the height of 1 m?
6. A big agency estimates that 60 flats of a big apartments complex can be rented if the rent is 600 euro. Moreover it estimates that for each 30 euro increasing the rent of 3 flats is lost. What amount must be asked in order to maximize the profit? What amount to get an income not less than 90% of the maximum profit?
7. In a movie multicinema 2500 tickets are sold every Saturday night if the ticket costs 5 euro. A marketing study estimates that for each increasing of 50 cents 100 less tickets are sold. Determine:
    a. The ticket price that allows to maximize the income;
    b. The maximum income;
    c. The number of sold tickets when the profit is maximum;
    d. The ticket price that allows to have a profit not less than 85% of the maximum profit.
8. A school has 1600 m of metallic net to enclose three equal rectangular adjacent (i.e. they have a side in common) game fields. What are the dimensions of the base (B) and of the height (H) of the rectangles that allow to maximize the area?
9. A mass is thrown up from the earth with a starting speed of 5 m/s. Considering only the gravity effect (hence neglecting the friction), whose modulus value can be approximated with 9.8 m/s^2, after how much time does the mass reach the maximum height? What its position after 0.1 s? After 0.4 s? What the mass velocity after 0.1 s? After 0.4 s? After 0.5 s? After 0.7 s? What the velocity when the mass falls again to the earth?

Activity 19 (homework to be discussed in the classroom)

The stopping space of a car: read and complete.
The stopping space of a car is determined by two quantities: the space run before the brake, given by the speed time the reaction time, and the braking space. Of course these quantities depend on the condition of the street, of the environment, of the car etc.
Let us suppose that we travel with a car at a constant speed of v m/s and that the reaction time is 0.8 s (generally the reaction times are between 0.5 s and 1.5 s).
The space run before starting the brake is given by ....
Let us suppose that the deceleration of the car is 9 m/s^2 and that the asphalt conditions diminish the acceleration of 30% (hence the mean deceleration is...)
Because of these data, the braking space can be calculated using the two laws of the uniformly accelerated motion (with $t_0 = 0$) :

$$s = v_0 t - 3 t^2$$

$$v_f = v_0 - 6t$$

But $v_f = 0$, since we are interested to stop the car; moreover is $v_0 = v$. Hence:

$$t = v/6$$

and consequently

$$s = v^2 / 12$$

Hence the total braking space is

$$s_{tot} = 0{,}8\ v + v^2 / 12$$

In a TI-nspire spreadsheet represent:
a) in the first column the speed in m/s (from 5 m/s to 50 m/s);
b) in the second column the speed in km/h;
c) in the third column, the total braking space;
d) in the fourth column the first differences of the spaces;
e) in the fifth column the second differences of the spaces;

Finally comment the table got in the spreadsheet: you can help yourselves reading the paper "La percezione del rischio e il rischio della percezione", by Franco Taggi e Pietro Maturano (http://www.matematica.it/paola/Analisi/spazio%20frenata.pdf) .

In the books for prospective riders it is written that the total stop space can be approximatively calculated dividing the speed in km/h by 10 and squaring the result. Is this estimate coherent with the data you got in your spreadsheet? How can you get it from the formulas you used in the activity?

c) The path towards calculus (second year)

In this part it is summarised the path that has been done in the last two months of the second year, which is an approach to the first ideas of Calculus. Most of the activities have been videorecorded.

Scheda 1 (= Worksheet 1). We work on local rectification of curves, as a cognitive root of the local linearity. The experiences are essentially of a perceptual-visual nature: the students are asked to use the zoom-on function of TI-nspire to observe the local rectification phenomenon. The function

“tangent” is used in order to have a control tool on the local rectification. The function “slope” is used to determine the slope of the tangent line and to have a more sophisticated tool to appreciate the differences between the tangent line and the linearized graph (got with 6-7 zooms on).
Scheda 1 ; Scheda 1bis ; Scheda1tris
TI-nspire file: analisi01

Scheda 2
It underlines that the local approximation in a point is different from the approximation in an interval. We expect that the students recall some experience about linear regression.
Scheda2
TI-nspire file: analisi02

Scheda3 ; Scheda3bis
The students are asked to determine numerically through a spreadsheet, the slope of a tangent in a point to a function (of course it is an approximation). Then the students are supported for passing from the numerical to the symbolic register.
TI-nspire file: analisi03

The following worksheets make the students to pass from a local point of view (the slope of the tangent in a point) to the slopes function (the derivative). We start with the function y=x^2: the students are asked to imagine the features of the function of its slope (the derivative). It is interesting to observe if the students use a graphical, numerical or symbolic approach. They are free to use Ti-nspire or, if they like it better, paper and pencil. Successively they are explicitly asked to pass to the graphic and then to the symbolic register.
Scheda 4; Scheda5; Scheda 6
TI-nspire file: analisi03 analisi04 .

Scheda 7
The inverse problem is approached. The idea is to pass from a function to one of its primitives. The students are asked to focus some of the properties and regularities that link a function to its primitives, particularly for polynomial functions. Also here it is interesting to observe if the students use a graphical, numerical or symbolic approach.
Scheda 7

4. Assessment

From the point of view of the teaching/learning practices in the classroom, the achievements of mathematical knowledge and competences by the students are highly positive.
We report here the results of a test given at the beginning of the first year to measure the starting levels of competencies (after two months of work in the classroom but before we started to work with TI-Nspire) of all the students of the school (only one classroom experienced TI-Nspire) and those of a final test, PISA-driven, made at May of the second year (again given to all the school’s students). The test is here given in the form given to the teachers, with all the indications for determining the scores. Of course the version for the students contained only the situations and the questions without the score indications. In another file we report the result of the final test: they suggest a significantly higher and more homogeneous level of acquired competencies from the students who experienced TI-nspire (class IIC).

Pretest con risultati e relativo commento (November 2006)
Test PISA (May 2008)
Confronto punteggi delle diverse classi seconde (comparison of the second classes).

5. Students' judgement about the handheld

At the end of the second year, the students have been asked to comment freely the qualities and defects of the handheld. The document limiti e potenzialità contains all their comments (in Italian), which are here summarised (in English). Some general comments on the handheld are in §3 of Part III.

PROS:
The handheld is a tool for learning
You have at hand a tool for answers and confirmations.
It allows explorations in graphic numeric and symbolic environments
You can save data
It bridges the gap between theory and computer; now mathematics is integrated with the calculator
It is very fast; you switch it on and in five seconds you can work.
Having learnt the Ti_nspire with the PC last year has been helpful for the approach to the handheld.
It gives the possibility of conjecturing in real time; any fresh idea is alive immediately in the software without the necessity of re-elaborating it for adapting it to the computer.
Good the possibility of transferring data to/from the PC; this improves also the visualisation of data.
It allows working in the classroom without having to go into the laboratory.
It helps in understanding where one is wrong.
The combination of keys can fasten the processes.
It is very robust.
It is possible to use it everywhere.
It gives the possibility of downloading the new versions of the software.
It is a communication tool between students.
It is a tool that allows to remain in contact with mathematics, keeping it subdivided in different environments but giving the opportunity to compare or unify them to make discoveries and checks.

WARNINGS:
You must use it critically.
With experience you can overcome such difficulties.
It can be useful if it is used with ingenuity.
It has no theoretical defects; but it has some practical disadvantages.

DEFECTS:
The keys are too small; but with experience one improve her/his abilities with the keyboard.
The mouse is too complicated and it is very difficult to grasp the objects on the screen using the cursor: this slackens problem solving.
The mouse is too small.
There are too many keys and they are too small.
The cursor is very slow.
The alphanumerical keyboard makes difficult writing hence the Notes environment is not used.
The elaboration of data is too long: for complex situations it may require hours.
There are many bugs.
There are errors in the regression line when it is approached in a modality different from MOD3.
The screen is too small: this makes difficult to study functions.
There are no colours
The handheld is too big.
It is difficult to build figures in G&G

In case such defects are eliminated it would be the prefect calculator.

SUGGESTIONS:
To use a touch screen.
To add a keyboard similar to that of the PC.
To fasten and make more sensible the cursor.
To eliminate some keys and substitute them with graphic windows.
To use a keyboard like the cellular phones.
To use a mouse like that of the laptops.

Some students gave marks to the handheld; here they are:
Software: 7 (over 10)
Calculator: 7
G&G: 9.5
Spreadsheet: 8
Notes: 4
Handheld: 7+

PART II
The classroom of P. Accomazzo

Approaching Calculus with TI-nspire (IV and V year of Liceo Scientifico; 17-18 y.o.)

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Corso PNI

1. The classroom and the students; the topics; the methodology

1.1 2006-2007: first year of experimentation

The class IV A PNI is composed of 25 students (7 girls). In this class the teaching of Mathematics is separated from that of Pysics. All the students are interested in scientific disciplines; they are clever students but are generally not so happy with the classroom routine. They accept to spend time to acquire a technique if they think there is really a reason for it. They have the habit of using different informatics tools both individually and in the laboratory; in particular they are acquainted with the TI-89 calculators.

The teacher has a good role with them, even if they do not accept a priori what she proposes but only they think it is convincing.

So the experimentation of TI-nspire has been inserted in such a human environment, rich of potentialities but not so easy to manage. The didactical intervention has concentrated on the function concept, with these specific goals:

a) To organize and interpret phenomena that appear different and that concern the concepts of variation, variation speed, local approximations;
b) To acknowledge, unify and formalize them;
c) To acquire the mathematical techniques necessary to formalize them;
d) To extend the set of known functions and the methods for studying them;
e) To widen the competencies in using functions for modelling different phenomena in different environments.

The experimentation has developed in two moments (only PC TI-Nspire in this first year).

First phase. November 2006-January 2007.

The students have become confident with the software using it

- to solve geometrical problems;
- to study exponential and logarithmic functions;
- to construct new functions from known ones and to study them from a global point of view (using the interplay between the algebraic and the geometric registers);
- to use the language of orders of magnitude to treat functions;
- to model problematic situations;
- to solve problems of maximum and minimum.

Second phase. February-May 2007.

Using TI-Nspire to tackle the following topics:

- The concept of linear approximation of a function in a point. The tangent line to the graph of a function. The slope of the tangent. The transition from the algebraic register to the computations of orders of magnitude and the "rules of Leibniz". The equation of the tangent for some classes of functions (polynomials, rational functions). The derivative function.

The experimentation has developed with 15 PC in the laboratory of theschool; each student had a copy of TI-Nspire at home. The moments of joint discussion and systematization have been done in the usual classroom space, where the teacher could use a projector connected to a laptop.

1.2 2007-2008: second year of experimantation.

In the second experimentation year there were only 20 students, since some of them were not admitted to the 5th year. The experimentation has involved also the handheld. Also in this year the experimentation has been divided into two phases: in the first short phase the students have integrated the practices with the handheld with their work; in the second longer one some important topics of Calculus, typically the Area problem, have been tackled using the potentialities of TI-Nspire.

First Phase: September-October 2007

Learning how to use the handheld solving problems about functions: the exponential functions, the number *e*.

Second Phase: November 2007-March 2008

The problem of the area below the graph of a function:

- the exhaustion method for the area of a parabolic segment;
- the method of rectangles for any polynomial curve.

The variation of the area behind the graph of a given function in an interval, while an extreme of the interval varies; the area function.
Problems about the study of functions.
Problems of maxima and minima.
Composition of functions: domain, zeroes, monotonicity, derivative.

1.3. The methodology

The methodology was in continuity with that of the previous year and followed the philosophy of the Laboratory of Mathematics suggested by the UMI proposals[3]. Each activity session started in the classroom: the students received a sheet with the text of a problem and they had to read it carefully and individually. Then a short collective discussion clarified the task and discussed the first ideas for a solution; in this phase the teacher aimed essentially to define the task and to facilitate the communication between the students.
Then the students divided in pairs (the pairs were formed spontaneously according to the students' choices). They found in their handheld a TI-Nspire file with the task and the sheets were they were supposed to work. The task asked that they explored the problem using TI-Nspire, showed what they did in the various worksheets and reported verbally what they had done in the Note-sheet. Sometimes they were also asked to produce a paper report to accompany the file.
At the end of the work the teacher collected the files of each group; the same happened with the home works: the students made their exercises and sent them to the teacher by e-mail. Each solved task ended with a general discussion, where starting from the works of the students the full story of the research were made, clarifying the main points, rephrasing the conceptual points, institutionalising the acquired knowledge, making explicit reference to their book.
The *time* spent in the classroom for the experimentation is about 30% of the full amount of hours for the mathematics curriculum (five weekly modules of 55 minutes); one can estimate that globally each student has spent 40 hours at home using the software.

[3] http://umi.dm.unibo.it/italiano/Didattica/ICME10.pdf; http://umi.dm.unibo.it/italiano/Didattica/didattica.html

2. The first year experimentation.

2.1 The appropriation and integration phase.

The had to solve some geometry problems. In the files (all the files are in the folder Pierangela) there are some of the exercises made by the students (using a version beta of TI-Nspire).

Spezzata

Triangolo

Angolo progressione

Farmaco

2.2 Algebra of functions and absolute orders of magnitude.

The major goal of the activities were to support students

- to improve their knowledge of trigonometric functions, included the graphics of sec^{-1} and tg^{-1};
- to evaluate the global behaviour of a function starting from its algebraic structure;
- to understand the absolute orders of magnitude (distinguished in three categories: very small, normal, vary large) and of the Leibniz rules to manage them.

We know that we can have a global (graph, formula) or local (coordinates point by point) perception of a function $y=f(x)$. Hence the students have been asked to explore point by point the function $y=1/f(x)$, and to compare it with the function $y=f(x)$.

The task was the following: the students had to make the software to draw the graph of the function $y=f(x)$, but not that of $y=1/f(x)$. Namely they had to focus on a point P of the graph of $y=f(x)$, then using the software they had to determine the coordinates $(x_P, f(x_P))$ of P and the value $1/f(x_P)$; finally they had to put the point $(x_P, 1/f(x_P))$ in the Cartesian plane and to observe its position with respect to the position of P.

Moving P on the graph they could observe the various positions of $(x_P, 1/f(x_P))$. At the end of thei explorations they had to draw on the paper what they thought to be the graph of $y=1/f(x)$, and finally they could ask TI-Nspire to draw the graph and compare the two to verify the correctness of their conjectures.

From the local explorations they had to point out the relationships between the two graphs: domain, monotonicity, signs, behaviour near a zero and near infinite (what happens to $1/f(x)$, when $f(x)$ is very large? When it is nearby zero?).

The idea is that you must build a trace of the function $1/f(x)$ with your mind's eye from the images you get from the screen. The worksheet in Graph and Geometry is simple to build and is general. So once the students have explored a function, e.g. y=sin(x) and y=1/sin(x), they can change the function of f1(x) and repeat the activity with another function. Concretely the students have studied y=sin(x) and then y=tan(x): it has been very interesting the study of the reciprocal of a discontinuous function.

Below (*fig.s 9, 10*) a screening of the exploration for y=sin(x) and of an answer for y=tan(x).

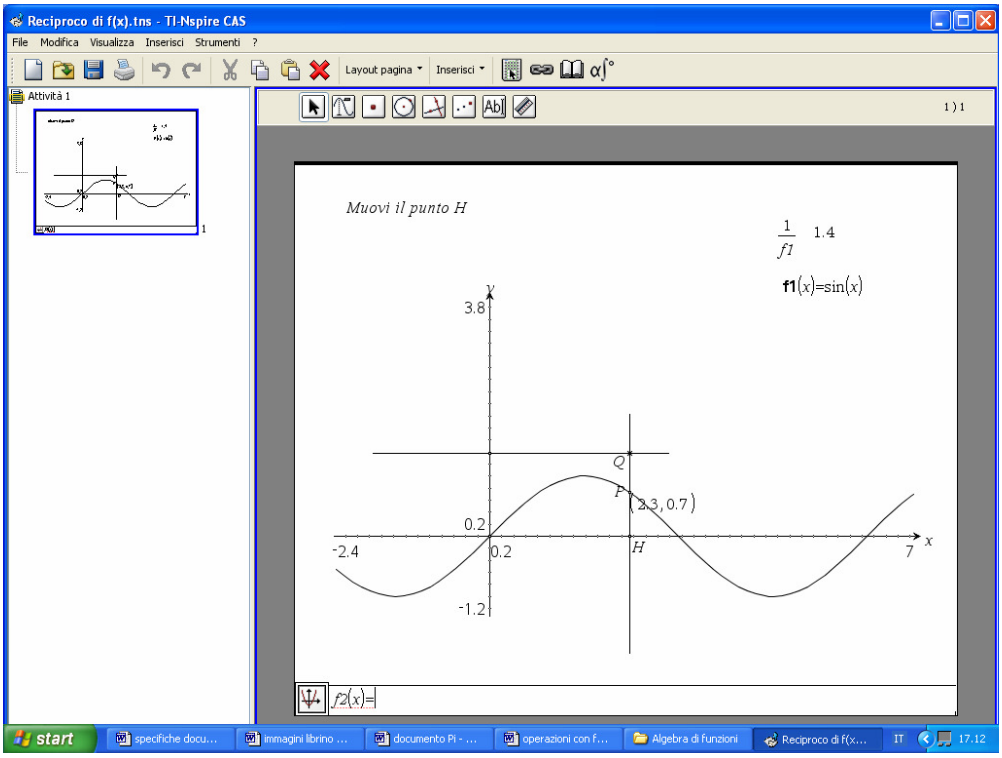


fig. 9

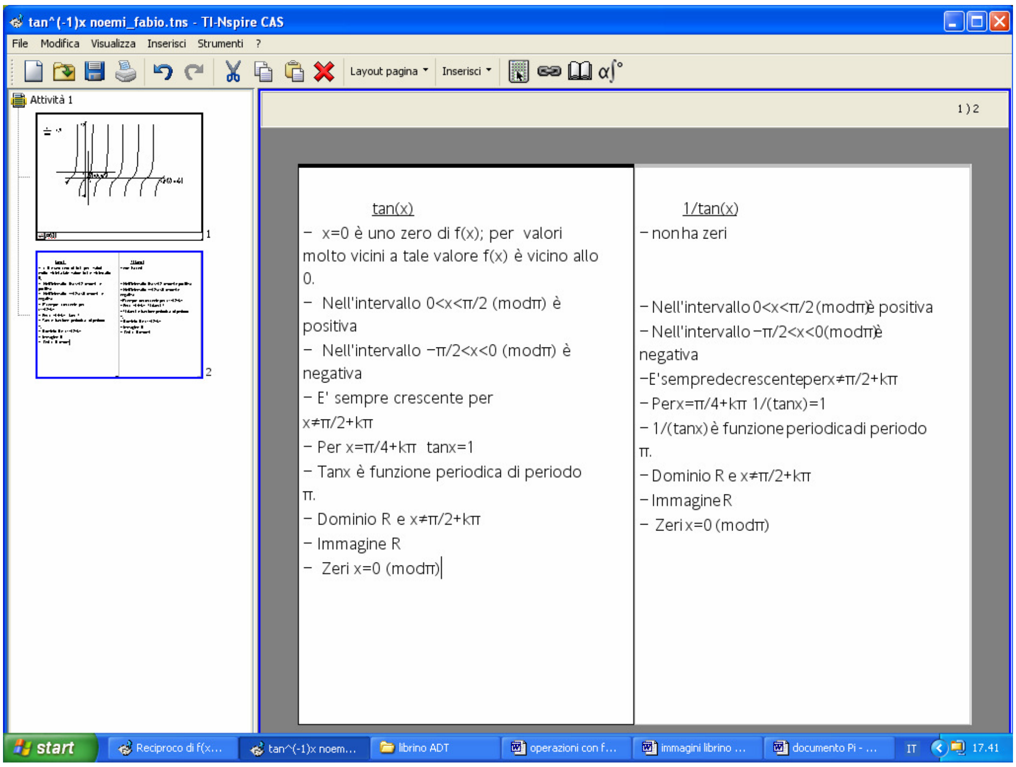


fig. 10

The activity has gone on with the study of the sums and products of functions (from $y = f(x)$ and $y = g(x)$ to $y = f(x) + g(x)$ and to $y = f(x) \bullet g(x)$). In the folder Algebra di funzioni there is a file .ppt with some answers of the students that have been considered by the teacher for the final discussion about the absolute orders of magnitude. The discussion has also been videotaped.

2.3 Tangent line to a cubic in a point

When students approached this activity they had already acquired a discrete familiarity with TI-Nspire and were able to move with easiness from one TI-Nspire environment to the other. They knew the set of absolute Orders of magnitude and their operative rules (reciprocal of an infinitesimal, of an infinite, of a normal; sums and products of infinitesimals, infinites, normal) and knew the cases of indeterminacy. Hence they were supposed to use their new knowledge in the new activity. Its aims were the following:

- to break the idea of global tangent line based on the notion of the tangent to a circle, which leaves all the curve to the same side; and to redefine the tangent as a local phenomenon, the tangent as the possible limit position of a secant;
- to get methods to determine the slope of a tangent (solving a system; calculated;...); to solve any problem where it is asked to determine the tangent to a curve from a point;
- to be able to evaluate the variations of the slope of the tangent to a curve while the tangency point varies on the curve.

The activity had different problems of increasing difficulty. The first two problems were the following:

Problem 1. The squirrel on the hill.

A squirrel stops at the foot of a steep hill, looking for a tree to climb. Upon the hill there is a high pile, that could be a good refuge for the squirrel. From its place can the squirrel see the pile? Follow the instructions in the file scoiattolo sulla collina.

In the file the skyline of the hill is a branch of the cubic $y = -x^3 + 4x$, the pile is a vertical segment from the maximum point of the cubic. At the beginning, the squirrel is in O. The length of the pile can be changed: the question is how high must the pile be so that the squirrel can see its top. In a second sheet the pile has a fixed length and the squirrel can move along the cubic: the students are asked to determine the position from which the squirrel start seeing the top of the pile. The visual line of the squirrel can be identified with the tangent line to the cubic where is the squirrel. The problem introduces the search for the equation of the tangent line in a point to a polynomial curve; at the beginning the point is O.
In this task the students could explore freely the figure and use some of the tools of TI-nspire (tangent, coordinates of a point;...) to solve the problem.
In the second problem they are explicitly asked to find algebraically the equation of the tangent.

Problem 2. Tangent to a cubic.

Determine the equation of the tangent line to the cubic of equation $y = -x^3 + 4x$ in its point of abscissa 1: how can you do? Open the file "tangente in un punto a una cubica".

Some students have tried to solve the problem using the same method they had used for the parabola (a pencil of lines through the point, then the system of its equation and the cubic and tentative of putting the delta equals to zero), but they obtained an equation of the third degree, which they were not able to solve. Other students have tried to build the equation of the line determining two of its points (*fig. 11*).

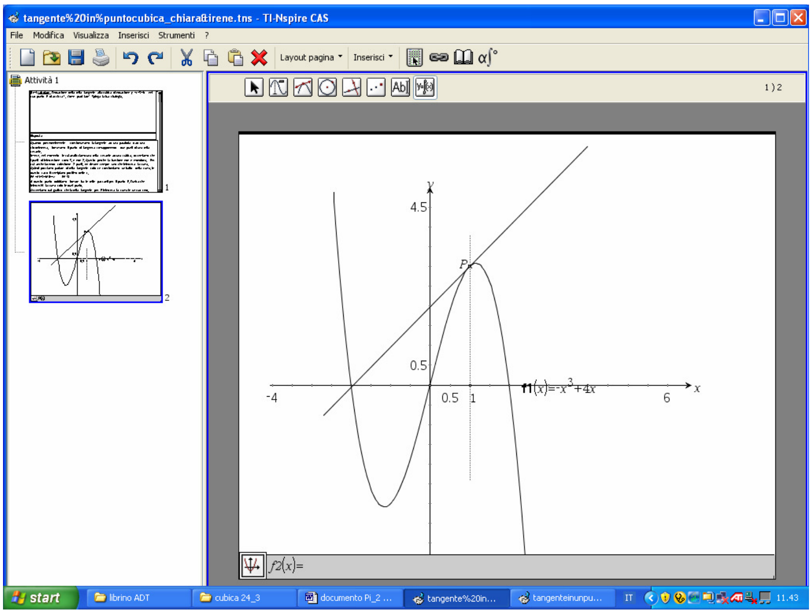


fig. 11

Here is a protocol of a student:

When previously I considered the tangent to a parabola or to a circle we found the point of tangency overlapping two points of a secant. But when we analyse the intersection points are 3 and not 2. This happens because the function is not monotone. So that even making two points to coincide, a third one remains that intersects the curve, in this case in the negative half plane of the x. P(1; -1^3+4·1)→P(1;3).We observe from the graph that the tangent through P intersects the curve in one of its zeroes. To find the slope of the line we take another point of this one (0,2) and make the ratio Δy/Δx which is 1.

The student has not checked the strategy: it should be enough to draw with TI-Nspire the line of equation got either checking that its zero was a zero of the cubic, or checking algebraically the double intersection of the line with the cubic.

A quality jump happened when some groups thought to find the equation of the tangent using a point and a very nearby other point; after some trials using numbers (e.g. P(1,3), Q(1.01;3.0097)) someone has thought to introduce a generic increment h. Here is the protocol of a student.

Il salto di qualità si è avuto quando alcuni gruppi hanno pensato di trovare l'equazione della retta tangente lavorando sul punto dato e su un punto molto vicino; dopo alcuni tentativi fatti con dati numerici (es P(1; 3) e Q(1.01; 3.0097) qualcuno ha pensato di introdurre un incremento generico *h*:

Determine the equation of the tangent line to the cubic of equation $y = -x^3 + 4x$ *in its point of abscissa 1*

*- first I determine the ordinate of P substituting 1 in the equation of the cubic: y=-1^3+4*1=3, hence P=(1;3).*

- before looking for the equation of the tangent it is necessary to say that we must consider only one branch of the cubic. In fact a line cannot be tangent to a whole cubic since its image is the whole R. Hence I consider that the line is tangent to the cubic in the branch 0≤x≤2.

- Since I have no mathematical law that allows me to calculate directly the slope of a tangent to the branch of a cubic I reason on the very small...that is I introduce a point K very near to P with abscissa x=1+h, where h is an infinitely small number. I find the ordinate of K substituting its abscissa in the equation of the cubic: y=-(1+h)^3+4(1+h)=(-h^3-3h^2+h+3)...The point K has coordinates (1+h;-h^3-3h^2+h+3). Since h is infinitely small, so are h^3 e h^2, hence the ordinate of K is very very near to that of P (a bit larger). Hence I can consider that the point K coincides with P. In such a way finding the slope of the line through the two points I find also the equation of the tangent to the curve through 'P coinciding with K'...: m(line)=(yK-yP)/(xK-xP)=(h-h^3-3h^2)/h...Since h^3 and 3h^2 are much smaller than h they can be neglected: hence the slope is a

number very very near to 1 and I consider it equal to 1. Now I easily get q(line)=2 and I can write the equation of the line... y=x+2.

From the students' protocol it is clear the break with their previous ideas about a tangent line, which strongly depended on their practices in geometry (line whose distance from the centre of the circle is equal to the radius) or in algebra (putting the discriminant of a second degree equation equal to zero). They are re-defining the notion of tangent as a straight line through two very near points and are so elaborating a general method, which can substitute or join the old algebraic one. The teacher suggests using the environment Calculator of TI-Nspire, to do easier the computations with the increment h; with the help of the teacher the students have computed the ordinate of P as *f*(1), have defined the variables *deltay* e *deltax,* have built the incremental ratio and have used the operator *limite* (the teacher has made them aware that such an operator works using the rules of Leibniz for the absolute Orders of magnitude).

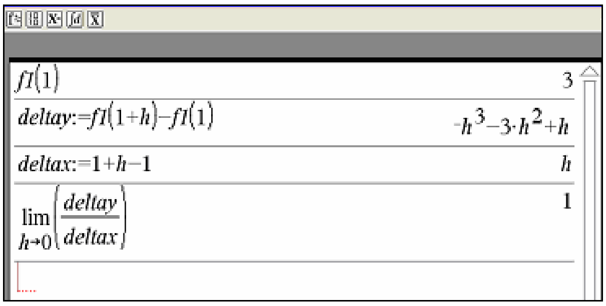


fig. 12

<u>Problem 3.</u> Describe how changes the slope of the tangent to a cubic in a point while its abscissa changes.
Open the file 'Descrizione variazione tangente'.
You can make TI-Nspire to find the slope of the tangent in a point and to draw it.
Can you evaluate how this slope changes while the tangency point abscissa changes?
Consider the function that sends the abscissa of the tangency point to the corresponding slope: can you imagine the graph of this function?

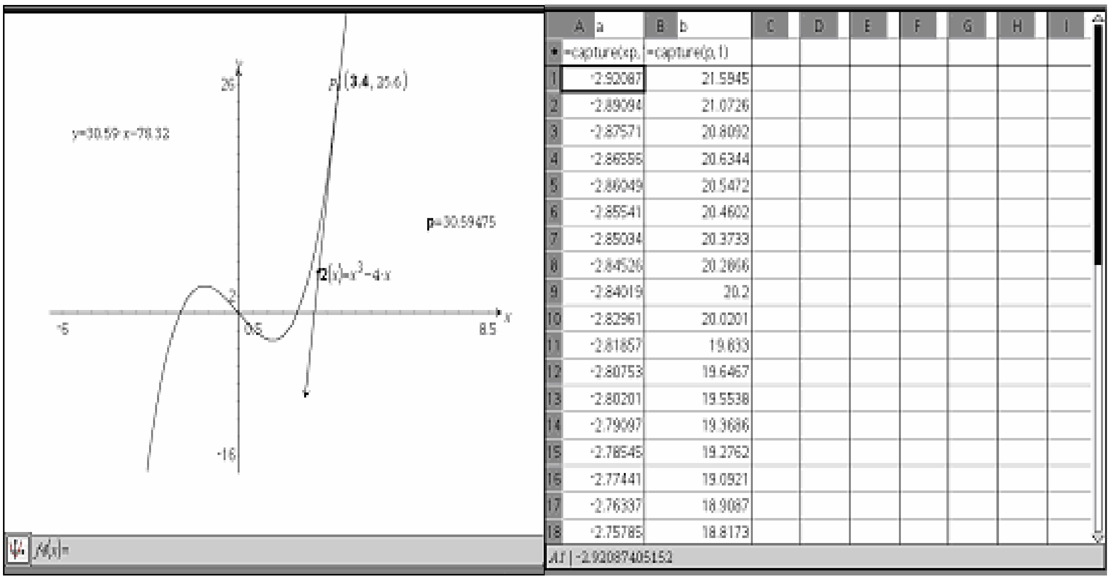


fig. 13

From students' protocols:

The values of the slope of the tangent in any point of the cubic depend from the behaviour of the curve in that interval: for the x of the cubic that tends to ±∞ the slope (p) assumes bigger and bigger values, near the two vertices it tends to 0, while for values of x near to the central zero, the slope tends to -4 (it is worthwhile observing that the coefficient of the first degree x in the cubic is equal to the coefficient of the x (p) in the equation of the tangent in the origin).

We have collected the values of xp and p while P varies on the cubic, we have activated the variables and in another graph we have represented the slope of the tangent line (p) in function of the abscissa of the point P on the cubic (xp). We have observed that the behaviour was near to that of a parabola of equation 3x^2-4; in fact drawing this function the curve passes through the points that have abscissa xp and ordinate p.

The starting cubic is odd (symmetry with respect to the origin) while the curve that represents the variation of the slope in function of the abscissa of the point P is an even function(symmetry with respect to the ordinate axis).

The vertex of the parabola is the least value of the slope and, as we had already observed, is -4.

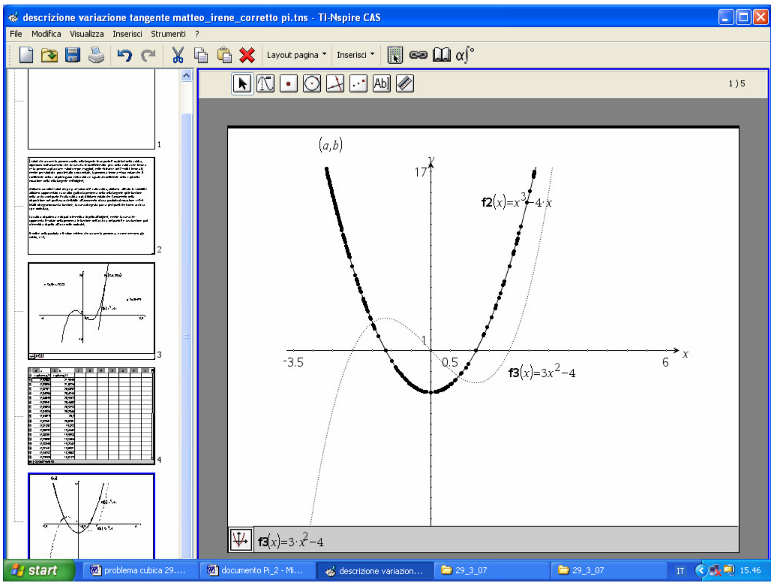


fig. 14

[Cubica](#)

Generally the students did not use so much the Calculator environment: they preferred to make calculations in paper and pencil or using the TI-82. The Calculator environment, with its strong symbolic features, has shown difficult for them: the theoretical notions necessary to manage it were difficult for the students. Moreover the students had the impression not to control so much the computations in this environment. Only after they have acquired a certain mastery and control in the computations with the infinitesimals, they have started to use spontaneously the Calculator of TI-Nspire.

2.3 Testing the work

In May a test was given to the students: the test was individual and each student worked at a PC: *Verifica*.

Products asked to the students: paper and file;

Time: 45';

The test measured

a) the level of:

- knowledge of TI-Nspire (Graph & Geometry, Spreadsheet, Calculator);
- understanding of the new definition of tangent;
- understanding the meaning of 'incremental ratio' and of the variables that appear in its formula;

b) how the students are able:

- to use the tools that TI-Nspire allows to determine the slope of the tangent to a curve in a point and the slope function;

The results have been quite good.

Moreover the students have been asked to fill a questionnaire to assess their experience with TI-Nspire.
Questionario
A detailed report on the questionnaire are in Intervista Ti rivista, Intervista Liceo Einstein
Here we make a summary of them:

1. The students like TI-Nspire more than Cabri (14 students over 23) and more than graphic calculators (18/23) that they had use systematically.

2. According to them the functions of TI-Npsire they like more are: Graphics & Geometry(18/23), Data capture and Spreadsheet (13/23).

3. The most useful features are:
    - The capability of TI-Nspire of allowing to work in different sheets and the possibility of linking them without transferring the data;
    - The possibility of capturing and manipulating data for drawing graphs;
    - The possibility of opening more working sheets of the same type but independent each other;
    - The possibility of organizing the work in an easy way;
    - The possibility of producing a complete and precise work;
    - The possibility of studying complex phenomena.

4. The most difficult functions is the Calculator (15/23); for some of them (7/23) also the Spreadsheet presents some difficulty.

5. 22 students over 23 say that TI-Ns helps; only one said: "sometimes".

An interesting question was the following: *Do you think that TI-Nspire can help you in experiencing genuine mathematical concepts that you must study or do you think the understanding such abstract concepts as the mathematical ones is relatively helped by the use of this software?*
Here the answers:

1. Yes, provided you are able to use its functions. Above all you can have immediate results, insofare one has not to spend a lot of time in long calculations that may facilitate errors.
2. It is useful if the work is guided and there is the possibility of discussing the results in group.
3. It helps to make interesting and useful experiences for studying some specific topics and for fostering a new working method.
4. Yes, since it is not a purely mnemonic study: through the exploration of the proposed problems, the mathematics can be remembered more.
5. It allows to have a practical idea of what one is studying.

3. The second year of experimentation

In the second year of experimentation the students have been introduced to the use of the handheld: each students received an handheld to be used at school and at home. It required some time to become expert in its use, because of the complex managing of the keyboard and of the limited capability of the screen, compared with the TI-Nspire on the PC.
At the beginning the students refused it. In order to foster them to use it the they were proposed to solve simple problems and small example of programming activities.:
Montecarlo

Bisezione

Then when they were able to use the handheld without too many difficulties we started to study exponential functions like $y = a^x$ in order to arrive to feature the base e; then we defined e as the limit of a specific succession. In this case they all worked with the handheld (only the task has been given on paper).

Numero *e*

Successively they tackled the derivation of trigonometric functions and then how to model an oscillating spring. With a CBR the time-space data of an oscillating spring have been collected. Using them the students have been asked to find the instantaneous velocity of the spring (in fact of a small mass oscillating with the spring).

The task suggested different research modalities for interpolating the speed function: the students produced a variety of solutions.

Derivate di sinusoidi

Molla

The first written assessment at the end of the first trimester was done in a traditional way, without using the handheld (the students at the end of the fifth year have a final examination where no computer or calculator is allowed). The teacher has discussed the correction of the assessment task with the students involving them in a wide discussion.

Compito 14_11_2007

Correzione compito 14_11_2007

We have faced the problem of determining the area between the graph of a parabola and the x axis in a given interval ([0,1] at the beginning.

The students have been divided into two groups: some used the PC some the handheld. The task is given so to guide the students to solve the problem in 1 hour.

Area sottesa da parabola

We have not observed a difference between the students using the PC and those using the handheld. Both alternated the different environments of TI-Nspire to solve the problem. At the end the teacher using view screen has explained the exhaustion method of Archimedes to determine the area of the parabola segment.

Segmento parabolico ed esaustione

To better evaluate the differences between the use of the Handheld and the PC the following open problem has been given to the students.

The bath-attendant problem

A bath-attendant is on the beach at point B and sees a bather in the see at a point A, who has serious troubles. On the sand the bath-attendant can run at a speed of 5 m/s while in the water he swims at the speed of 2 m/s; the border between the sand and the water is perfectly right.

In which point P must the bath-attendant dive so that he can reach the bather as soon as possible? Justify your answers.

We had discussed with the teacher the better way to give the task. At the beginning we thought it better to give a file to the students where the points A and B were blocked; we foresaw that the students would have solved the problem in two ways:

- numerically, with data-capture;
- algebraically, building the time function according to the variation of P.

Consequently we were waiting for solutions like the following ones:

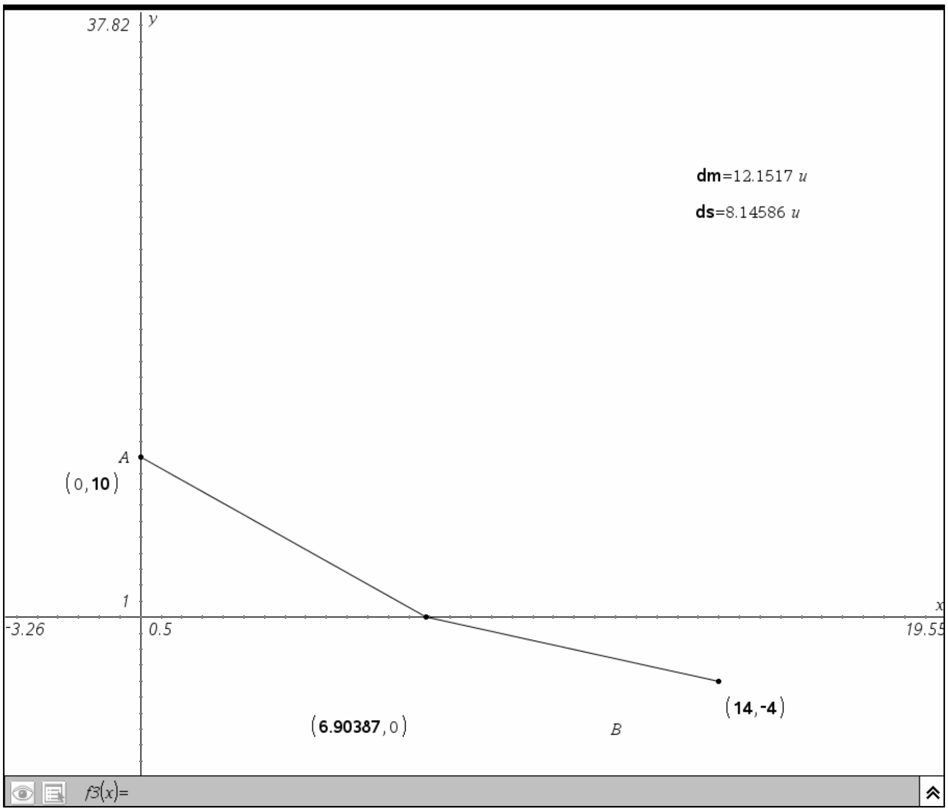


fig. 15

	A xx	B	C	D tt	E	F	G	H	I	J	K	L
◆	=capture(xp,1)	=capture(dm,1)	=capture(ds,1)	=b[]/2+c[]/5								
1	.671823	10.0225	13.9155	7.79436								
2	.40663	10.0083	14.1697	7.83807								
3	.362431	10.0066	14.2121	7.8457								
4	.318232	10.0051	14.2545	7.85343								
5	.274033	10.0038	14.2969	7.86126								
6	.229834	10.0026	14.3394	7.86919								
7	.185635	10.0017	14.3818	7.87722								
8	.141436	10.001	14.4243	7.88536								
9	.097238	10.0005	14.4667	7.89359								
10	.497238	10.0124	14.0828	7.82273								
11	.897238	10.0402	13.6997	7.76003								
12	1.29724	10.0838	13.3177	7.70543								
13	1.69724	10.143	12.9367	7.65884								
14	2.09724	10.2176	12.5569	7.62016								
15	2.49724	10.3071	12.1784	7.58923								
16	2.89724	10.4112	11.8013	7.56589								
17	3.29724	10.5296	11.4258	7.54995								
18	3.69724	10.6616	11.052	7.5412								
19	4.09724	10.8068	10.6801	7.53943								
20	4.49724	10.9647	10.3103	7.54442								
21	4.89724	11.1348	9.94285	7.55595								
22	5.29724	11.3164	9.578	7.5738								
23	5.69724	11.5091	9.21607	7.59774								
24	6.09724	11.7122	8.85741	7.6276								
25	6.49724	11.9254	8.50244	7.66317								
26	6.89724	12.1479	8.15164	7.70429								

C265

fig. 16

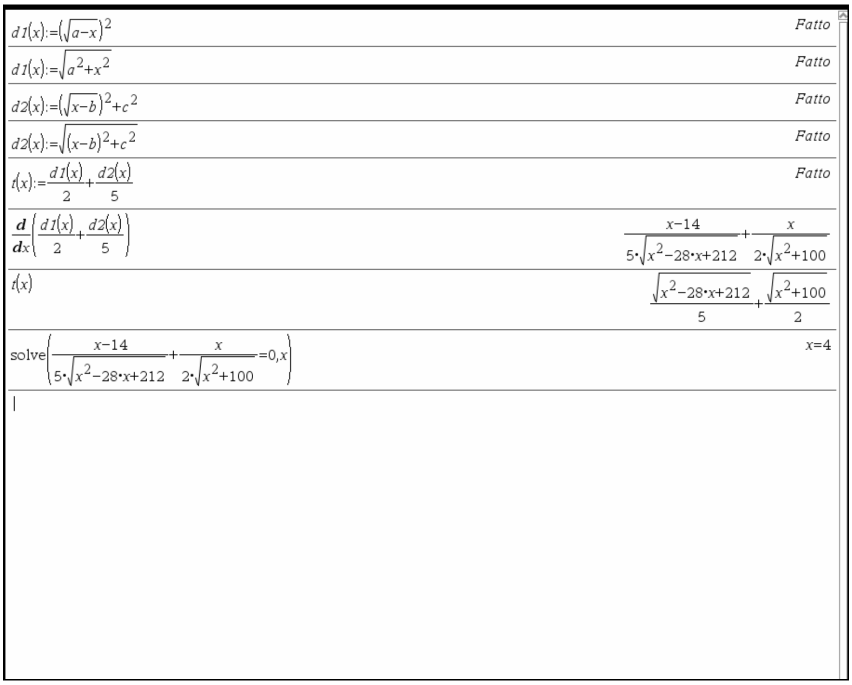


fig. 17

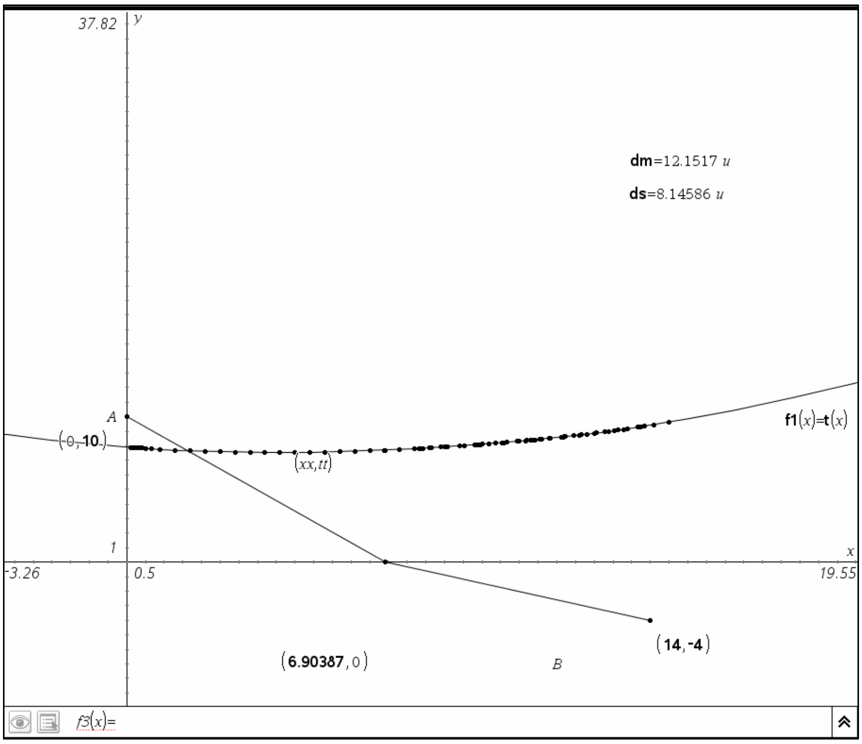


fig. 18

But then we preferred to give them the task on a sheet of paper with the two points A, B in a generic position and to leave them the initiative to build its representation in TI-Nspire as they liked better: the reason of this decision was that we did not like to suggest immediately the analogy with refraction problems that they had studied in Physics.

In the first phase of their activity the students worked to the interpretation of the text. Those who tried to find some connection with Physics did not think to refraction but to weird connections with speed and acceleration. Only a few thought immediately to represent the situation in a Cartesian plane: the groups who had chosen to work at the PC have started soon drawing the situation in the G&G environment, while those who preferred to work with the handheld chose to mimic the situation moving themselves on the floor of the classroom. Moreover while those with the PC worked as usual in pairs, those with the handheld spontaneously started a common discussion on the possible solutions to the problem. They started to use the tool only after the teacher suggested them to represent the problem in the G&G environment. Both types of groups in any case then used the spreadsheet and the data capture to get the solution.

In the solutions we observe:

- differences in the positions of A and B within the Cartesian plane (not all have drawn the simplest situation);
- differences in the use of TI-Nspire tools, like data-capture or the Calculator.

In fact some have explored the problem remaining at a purely geometric level: they have dragged the point P, observed what were the limit positions for P (perpendiculars to the line r from A and B) and have checked that the least distance between A and B did not correspond to the least time.

Those who have used the spreadsheet have introduced the formulas to get the time: e.g. they put in C column the distance in the see/2 (dist_mare), in D column the distance on the sand/5 (dist_sabbia) and in E column the sum between the previous two.

Using a scatter plot they have then got a graph of the time function and asked themselves what curve it was. Unfortunately they had only 50 minutes to make the task and they were not able to answer the question during that time: they made this part at home individually.

Hence we can observe that in cases where the situation is free and students are not guided so much in their work:

- the approach to the tool depends on the problem formulation;
- within the same problems the students who use the PC use the tool to explore freely the situation, to conjecture, etc.: in this sense the PC TI-Nspire has revealed for them a real thinking tool; the students used it spontaneously as a natural prosthesis without any effort;
- on the contrary, students with the handheld use it only when they have an idea of the solution;
- it is not automatic that some tools of TI-Nspire are immediately present to the students for solving a problem; sometimes it is necessary that the teacher suggests them to use them; hence not only the T-Nspire and the Handheld are interiorized at different levels but this happens also for the different functions of the software.

Comparing these observations with what happened in the classroom of Domingo, it is evident a difference: these difficulties with the handheld were less evident in that classroom. The reason is not so clear, since it is not easy to compare the two groups because of the difference of age of style in the two teachers, etc. So it is not clear if this happens because of a poor instrumentation with this part of the software, or for some didactical reason specific of the classroom, or for some other reason. This point is worthwhile of further investigation.

A possible conjecture is that the students of Pierangela were already acquainted to use other calculators, typically TI-82, from the beginning of the secondary school; they did not like the different and more difficult keyboard of the Handheld. On the contrary the students of Doming had no previous experience with other calculators and started with handheld from the beginning, so had no prejudice about it. But this is a pure speculation.

We allege two files: one with the problem solved with the PC (bagnino fabio_roby) and one with solution with the Handheld (Irene_matteo).

After the students had completed the solution at home, there has been a classroom discussion; it has allowed to point out the following points:

- The relevance of the system of reference;
- The limit positions of P;

- Analogies and differences between this problem and another simpler problem about a least distance problem, given some weeks before

The bath-attendant problem has been the last activity considered for the experimentation. Later the teacher has systematized students knowledge because of the final examination the students had at the end of the year. The students have use the Handheld individually more than before. TI-nspire has become for them an everyday pragmatic tool, useful to make calculations, visualize and study functions, check results got in paper and pencil environments. Moreover the usual communication through e-mail between the teacher and the students has become an 'online help', which massively used TI-Nspire. The students asked questions and many times the teacher answered sending them TI-Nspire files that could help them to answer themselves their questions.
This communication tool between the teacher and the students is new and very; in these last years such fresh ways of communicating are widely studied (see e.g. Borba & Gadanidis, 2008). Within TI-nspire environments they assume a specific connotation because of the multirepresentational features of the software: see the comments in part III, §3.2.

4. A lecture given by the students

The students have given a lecture to mathematics teachers about their experience (Conferenze Mathesis: 24.11.2007). See: 'Presentazione Mathesis'.

5. The examination task

On June 19, 2008 the students have had the final mathematics examination. The file Y557 contains the task they had to do (without calculators or PC).
Eighteen students choose the problem 2 and two the problem 1. The least answered questions have been number 5 and 6; the most chosen was number 10 (a topic very well developed in the last three years)
All the students have done well their task, using a big variety of solution modalities. It is interesting to observe that when they did not remember a formula to treat symbolically a task, they used numerical and graphical methods with success. As an example, in question 4 of problem 2, thay had to find the area of $y = 2^x - x^2$ in [2;4] and since they did not remember the integral of 2^x, some have drawn the graph of the function, has observed that the function is monotone increasing with concavity upwards and has found the area of the trapezium in the figure, observing that this is an approximation of the requested area. Namely those students were able to mobilise different abilities and competencies in order to face a problem.
The score got in the examination are very good: taking 10 as a sufficient score only 3 students have got a lesser score. The scores and the performances in the other class, where no experimentation had been done, were completely different (see the Tables 1, 2). Also in that case the students have generally chosen problem 2, but they got lost in many calculations, particularly in looking for the approximate solutions of an equation; moreover they have answered less questions than in the other class. It is a conjecture that possibly the experimentation with TI-Nspire has helped them in developing more those different competencies that have been useful to solve better the final task.

TI-Nspire CLASS

scores	abs. freq.	rel. freq.
7	0,125	2
8	0,125	2
9	0	0
10	0,25	4
11	0,125	2
12	0,1875	3
13	0,125	2
14	0,0625	1
15	0	0

# students	16
mean	10,5
mode	10
sigma	11,68

Table 1

control CLASS

scores	abs. freq.	rel. freq.
8	0,05	1
9	0,1	2
10	0,1	2
11	0,1	2
12	0,1	2
13	0,4	8
14	0,05	1
15	0,1	2

# students	20
mean	12
mode	13
sigma	1,48

Table 2

Some first conclusions about Pierangela experimentation

The students have been deeply involved in the research project and have worked hard to that. All the work proposals have been perceived by all of them as problems to explore and to solve. At the beginning they used more school oriented strategies (immediate tentative to formalize algebraically the problem, tentative to use the last topics explained in the classroom to solve the problem, even if it this was not the case, etc.). Little by little their approach has changed: first bashfully, then with more self-possession they have used a different approach, where strategies based on their perceptions of the data (graphic explorations, changing the parameters of the graphic), on the use of

automatic information given by the system (slope, data capture, locus, ...), on the connection between the different pieces of information given by the system. In the same time they had the habit of checking the trustfulness of the automatic answers of the system, with a parallel use of paper and pencil.
This change has been induced by TI-Nspire more than by the types of problems or by the teacher style of working. Its integrated environments support thinking 'associatively' more than 'sequentially', as usually requested in the traditional mathematics lectures.
Those students not so akin to the systematic traditional way of working in the classroom, but clever in managing multitasking situations, feel free to use their extra-school competencies and their usual ways of solving everyday problems by trials and errors; and this has given good results. The most diligent students have been more cautious in abandoning their traditional and safe way of working; but notwithstanding this they have acquired a good mastering of the tool. The low achievers in mathematics (4 of 25 in the first year and 3 of 20 in the second one) have shown similar difficulties using TI-Nspire.

PART III
A theoretical frame for TI-Nspire: how to evaluate a two years experimentation

The present report is based on a the analysis of the collected data (2006-2008), in particular from about 80 hours of videos of the performances of the students.
It is divided in three chapters: the first develops the general theoretical frame for TI-Ns CAS, within which the research has been developed; the second contains an elaboration of the data analysis, which pictures some of the main features of learning processes when using TI-Ns; the third deepen the analysis and points out some feebleness of the handheld, which seem possibly overcome with the introduction of the software TI-Connect-to-Class.

1. The general analysis of TI-Nspire Environment.

The introduction of technologies in the classroom generally produces a very complex situation. This is particularly true for CAS TI-Nspire (TI-Ns form now on), which is a very sophisticated software. Roughly speaking, the first major question (MQ1) when a new powerful but complex instrument (like TI-Ns) is introduced in a classroom is: *can it really support a better learning in students (through a suitable didactical design) and how? Which are its strong and feeble features?*
When considering the specificity of TI-Ns, a second major question (MQ2) comes up, based on the fact that the TI-Ns is available in two different environments: the PC and the handheld: *are there differences between the learning processes supported by the two different environments of TI-Ns?*

In this first chapter we transform these naïve questions into more scientific ones and formulate some hypothesis for some answers, based on the experimental studies. In the third chapter we give some answers, which are based on the two-years experimentations developed in two secondary schools, the first year using the PC the second using also the handheld. The second chapters gives some information about the classes where the experimentation was developed and on the sequence of problems that the students faced using TI-Ns. The fourth chapter supports some of the claims made in the third one summarizing some of the pre- and post- test given to the students of our experimental classes. A fifth chapter is not written but it is made of some videos, which illustrate and support some of the claims made in the third chapter. Some references conclude the report.

1.1 The general problem.

When working with calculators or computers, teachers and students are faced with two slightly different transpositive[4] worlds. One is the ordinary transpositive world associated with paper and pencil environments and the other is the computer environment.
Any research about learning processes in technological environments must be sensitive to the necessity of integrating the two aspects and to the related cognitive and didactic problems (see Artigue, 1997).
From the one side, we need theoretical frameworks that allow us to approach the institutional and cultural dimensions of such teaching-learning processes. From the other side, it is necessary to recognise the fact that teaching-learning mathematics in computer environments introduces a strong instrumental dimension to the corresponding processes. We speak of *instrumented actions*, insofar the actions of students are deeply ruled by the instrument's schemes of use: e.g. to compute the roots of an equation in TI-Ns, students can use the suitable function in the calculator modality. Both have strong consequences on the cognitive dimensions of didactic phenomena and these must be

[4] The *didactical transposition* has been studied scientifically by Y. Chevallard (1985) and by G. Brousseau (1997); roughly speaking, it concerns all the engineering necessary to transform a piece of mathematics so that it becomes learnable from a specific group of pupils. The transposition concerns not only the mathematics to be taught but all the environment, where its learning-teaching is designed: the teacher, the students, the used artefacts, the previous knowledge of students, and so on.

carefully considered.
Because of these reasons, we design our research within the following frames:

- the *anthropological approach,* developed by Chevallard (1999);
- the *instrumental approach,* coming from cognitive ergonomy of Rabardel and Verillon (1995);
- the *multimodal paradigm,* coming from cognitive science (see Wilson, 2000, for a summary).

Refining them we have elaborated a unitary model, the *APC-space*, where we can suitably formulate and face our specific research questions concerning TI-Ns.

1.2 The Anthropological approach

The Anthropological approach (indicated as ATD = Anthropological Theory of Didactics) assumes an institutional conception of the mathematical activity: Mathematics, like any other human activity, is something that is produced, taught, learned, practised and diffused in social institutions. It can be modelled in terms of mathematical *praxeologies*. These consist in the set of techniques and the related theoretical discourses that are linked to a precise task: e.g. the praxeologies for determining the tangent to a circle in a point or those for solving an equation concern both the usual techniques for performing such tasks and the related mathematical discourse that justify such techniques. Of course the praxeologies change according to environment: paper and pencil or a DGS software may support different praxeologies for the same task, and the praxeologies in Cabri may be different from those in TI-Ns.

Mathematical praxeologies are the object of learning and teaching in the schools: "Mathematical knowledge is produced, taught, learned, practised and diffused in social institutions. It is thus not possible to separate it from its process of construction in a specific institution" (García et al., 2006). Hence the cognitive processes, which happen in the classroom are institutionally pictured. This aspect points out a general problem, when introducing a fresh software (not necessarily TI-Ns) in a classroom; the praxeologies are supposed to change, but this does not happen automatically: the didactical design, the influence of the previous curriculum, the role of the teacher, the attitudes of the students with respect to the new software, the reaction of the colleagues and of the families, the general attitude of students with respect to ICT, etc. become important institutional variables to take into consideration.

Praxeologies must not be considered in an abstract way but in the ways they concretely appear in the classroom. In fact, any praxeology is always activated through the manipulation of *ostensives* and the evocation of non-ostensives (e.g. the ostensive "$y = a \cdot x + b$" and the mathematical abstract concept "straight line"), which are like the two sides of the same coin. This aspect of praxeologies is particularly intriguing in the case of some mathematical software (e.g. TI-Ns). In this case, some praxeologies are made palpable through precise actions which are necessary to activate the mathematical functions of the software: the ostensive part is represented by a button to push or by an operation to accomplish with the mouse. TI-Ns is plenty of such ostensives, some of which are really new and intriguing (e.g. the functions *calculate*, *data capture* and many others).

In the traditional mathematical activities generally the focus is on the non ostensives (the concepts), while the ostensives are underestimated. Things change when software like TI-Ns (or other software) enters the classroom: here the ostensive are particularly scrutinised, even if there is the risk to forget their non-ostensive mathematical counterpart.

The ostensives must be considered according two of their main functions. The first is a *semiotic* function: namely they are perceivable objects, which can be manipulated and can represent other objects (in such sense they are signs). But ATD points out another important, usually neglected function of ostensives: their *instrumental* function. The ostensives are not simple *working media* but genuine *instruments* for the mathematical activity: their careful manipulation does not only allow performing a mathematical task but is essential for its accomplishment. The instrumental and the semiotic value of the ostensive objects depend on the practises of the institutional system, where

they are activated. Consequently the non-ostensive objects exist because of the manipulation of the ostensive ones within specific praxeological organisations.
Rephrasing Q0 within this frame we get the following question

(Q1): *How praxeologies change in the classroom because of the introduction of TI-Ns*?
How the fresh praxeologies (if any) help(or block) students in learning mathematics?

In our case the comparison is not (yet) with a traditional classroom but with innovative teachers and in one case with students who just enter the secondary school (hence they do not yet practice the praxeologies of this order of school) and in the other case with students of the 12th grade, who already know different software (spreadsheet, Cabri) and have intensively used the TI calculators.

1.3 The Instrumental Approach.
P. Rabardel and others introduce the concept of *Instrumental genesis* to describe the process by which an artifact becomes an instrument. It indicates the two directions in which this process takes place: towards the self and towards outside reality. The first meaning of appropriation requires the artifact to be integrated within one's own cognitive structure (e.g., one's existing representations, available action schemes, etc.) that in general, require adaptation. Rabardel (and Verillon) termed this self-oriented construction "instrumentation." The second meaning indicates that the artifact has to be appropriated to an outside context. Specific ends and functional properties −some not necessarily intended by design− are attributed to it by the user. Adjustments are made to account for goal and operating conditions. Rabardel (and Verillon) called this "instrumentalization".
Fig.19 (taken from Maschietto & Trouche, to appear) illustrates this double arrow:

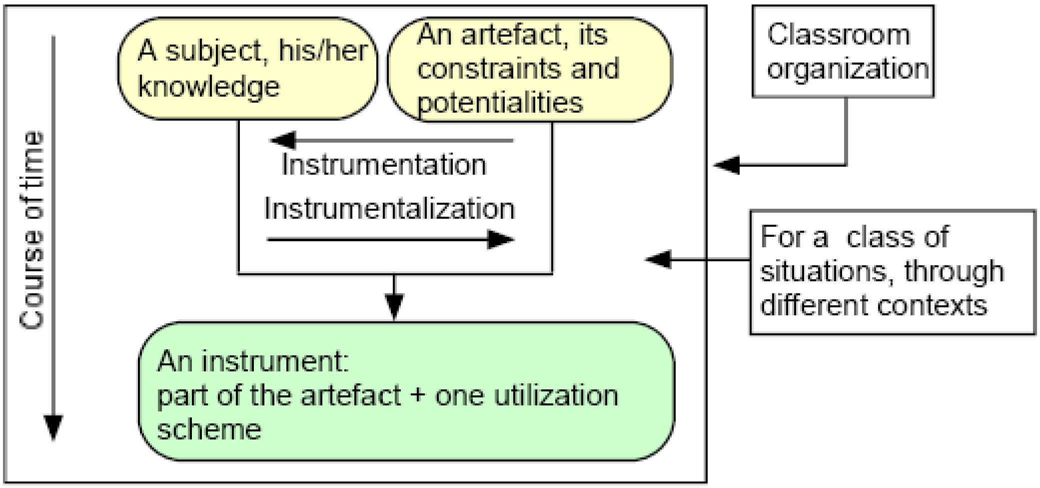


fig. 19

For example students learn to properly use data-capture in TI-Ns through an instrumentation process. But they generate an instrumentalization process, when they use the potentiality of random number generators in producing samples of data in the spreadsheet for solving a specific fresh problem.
Instrumented action can be broadly distinguished according to whether it aims at producing transformations (*pragmatic action*) or affording knowledge (*epistemic action*) [5]. Artifacts (such as sensors, meters, and computers) are also used to derive knowledge concerning the environment by detecting, registering, and measuring some aspect of reality not immediately accessible to the user.

[5] People working in cognitive science distinguish between pragmatic actions—actions performed to bring one physically closer to a goal—from epistemic actions —actions performed to uncover informatioan that is hidden or hard to compute mentally (see Kirsh and Maglio, 1994). In particular epistemic actions have been used by Herschkovitz, Baruch and Dreyfus (2001) in the context of mathematics learning. They define an epistemic action as a "mental action by means of which knowledge is used or constructed". Epistemic actions are often revealed in suitable settings, e.g. when students an instrument or when they interact.

E.g. students may refine their actions in TI-Ns for producing a more uniform set of data with data-capture through an animation or may decide to use data-capture to explore the relationships among the sides and an angle in a variable triangle, subject to some constraints.
Rephrasing Q0 within the instrumental frame we get the following issue

(Q2): *What is the specificity of instrumental actions in TI-Ns environment*?
What instrumented actions in TI-Ns help (or blok) students learning processes?

More specifically: what are its instrumentation/instrumentalization aspects, what the interplay between pragmatic and epistemic actions? How the activated instrumented actions can (or do not) support student learning?

1.4 Multimodality.

The third issue concerns *multimodality* and *embodiment*. The notion of *multimodality* has evolved within the paradigm of *embodiment*, which has been developed in these last years (Wilson, 2002). Embodiment is a movement afoot in cognitive science that grants the body a central role in shaping the mind. It concerns different disciplines, e.g. cognitive science and neuroscience, interested with how the body is involved in thinking and learning. The new stance emphasizes sensory and motor functions, as well as their importance for successful interaction with the environment. This is particularly palpable when observing the interactions human-computer. A major consequence is that the boundaries among perception, action and cognition become *porous* (Seitz, 2000). Concepts are so analysed not on the basis of "formal abstract models, totally unrelated to the life of the body, and of the brain regions governing the body's functioning in the world" (Gallese & Lakoff, 2005, p.455), but considering the *multimodality* of our cognitive performances.
Instrumented activity in technological settings is multimodal in an essential way. Many times it reflects the fact that action is not only directed towards objects but is also directed towards persons. Action in such situations can be seen as aiming at altering another subject's state of information. *Communicative action* is of this type, and subjects (teachers and students) simultaneously use a wide array of verbal, gestural, and graphic registers to communicate their thought. All such components or modalities (written signs, oral language, body enactments, artefacts use, etc) intervene in an intertwined way in learning and more in general in knowledge formation. A multimodal mode is typical of people using computer; but this is even more intriguing in the case of TI-Ns, because of its multifaceted environment, where different representations are simultaneously present. The multimodality of the used semiotic resources is amplified by TI-Ns.
Mathematics learning, as it happens in the described context, can be fruitfully analyzed through a semiotic approach that allows us to consider both its cognitive and its didactic dimensions. The resulting semiotic analysis therefore considers a plurality of semiotic resources that goes far beyond the written symbolic systems and oral language, to include gestures and bodily means of expression within the TI-Ns environment.
This frame allows refining Q0 as follows

Q3. *At what extent does TI-Ns modify the usual multimodal behaviours of students?*

The three questions Q1, Q2, Q3 can be investigated through a unifying lens, namely the *Space of Action, Production and Communication* (in short, APC-s), elaborated by Arzarello and his collaborators in these last years (Arzarello, 2005; Arzarello & Robutti, 2008). This model allows framing learning processes that happen concretely in the classroom according to the multimodal approach, taking into account the institutional and cultural aspects of the ATD paradigm and framing the variables pointed out by the Instrumental approach within a richer didactical environment.

1.5 The Space of Action, Production and Communication

The APC-s describes the processes, which develop and are possibly shared in the classroom among students (and the teacher) while working together. It analyses them considering their different components and a variety of mutually dependent relationships among them.

The components are: the body, the physical world, the cultural and institutional environment: in a word, the students themselves and the teacher along with the context where they are acting and learning. When students learn mathematics, these and other components (e.g. the emotional ones) take an active part in the learning processes, interacting together. The interaction comes from the students' work, the teacher's mediation and from the use of artefacts (TI-Ns in our case, but also their previous practices with the calculators and with Cabri in the case of the 12th grade students).

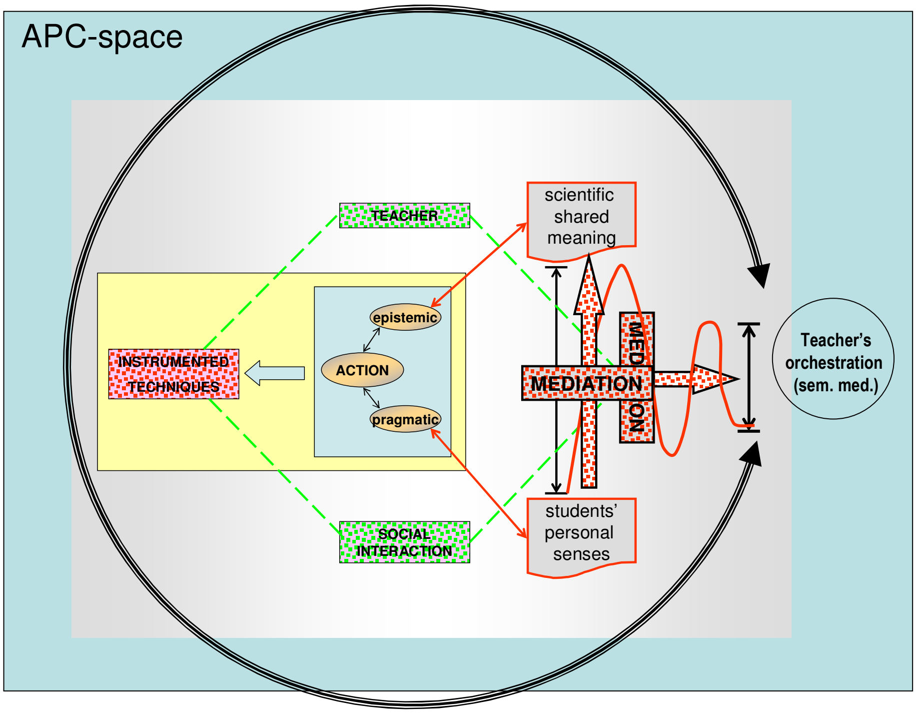


fig. 20

The three letters A, P, C illustrate the main dynamic relationships among such components, namely students' actions and interactions (e.g. in a situation at stake, with their mates, with the teacher, with themselves, with TI-Ns), their (individual or collective) productions (e.g. activating a function of TI-Ns, answering a question, posing other questions, making a conjecture, introducing a new sign to represent a situation, and so on) and communication aspects (e.g. when the discovered solution is written in the Notes of TI-Ns, communicated orally or in written form to a mate or to the teacher, or sent to the teacher through an electronic platform). The APC-s is a typical complex system, which cannot be described in a linear manner as resulting by the simple superposition of its ingredients. It particularly models how the relationships among its components develop in the classroom through the specific actions of the teacher. The APC-s components analysed through a semiotic lens allow making palpable the main features of the learning processes in technological environments. To grasp them, it is necessary to look at what happens while students are using TI-Ns, considering their multimodal productions (not only what they write) in a very detailed way. This is possible only through a careful analysis of the videotaped lectures. In general we can say that students start attaching their own personal interpretation to the signs they are exposed through the given task (e.g. when requested to solve a problem: see the examples below) and the teacher pushes them towards the scientific sense through a careful design of the didactic situation and through suitable use of the artefacts (we call this strategy *semiotic mediation*: see Arzarello & Paola, 2007). For an example, see the discussion in § 2.2 on the finite differences of second order functions. The dynamic interplay among the different components of the APC-s in TI-Ns environments are illustrated in *fig.20*.

Using the APC-s lens, the three questions above can be refined and investigated from a unitary point of view and some first answer can be put forward. These will be sketched in the following part of the report. All that follows is based on a first analysis of some of the videotapes and has a purely hypothetical value, which hopefully will be confirmed and better defined after the complete analysis of all the videotapes. The main issues will be the starting point to design the experimental study for next year.

2. Some answers to the first three research questions.

I recall here the three research questions discussed above:

Q1 *How do praxeologies change in the classroom because of the introduction of TI-Ns? How the fresh praxeologies (if any) help(or block) students in learning mathematics?*

Q2 *What is the specificity of instrumental actions in TI-Ns environment*? *What instrumented actions in TI-Ns help (or block) students learning processes?*

Q3. *At what extent does TI-Ns modify the usual multimodal behaviours of students?*

The major results that grow up from our analysis through the APC-s lens can be so summarised:

(i) A triadic structure seems necessary to describe properly the learning strategies of students who are using TI-Ns: *epistemic, (quasi-)empiric, pragmatic*. A consequence is the productions of fresh praxeologies in mathematics, which positively support learning processes of students. The new structure supports different tempos in the way students solve problems, compared with the typical paces with other software.

(ii) TI-Ns support specific instrumented actions, which push the students towards a meaningful use of symbols, namely new praxeologies are introduced through specific instrumented actions that positively support the treatment and the conversion of the symbols of mathematics (Duval).

These two major results allows to give the following answers to the three questions Q1, Q2, Q2:

Ad Q1. *New praxeologies are introduced in the classroom because of fresh specific instrumented actions supported by TI-Ns. Some of them have a positive consequence on learning processes of the students and on their attitudes towards mathematics.*

Ad Q2.*The major new entries in the instrumental actions supported by TI-Ns concern the specificity of the transition to the theoretical side of mathematics, to its modeling and to a meaningful introduction to the use of symbols.*

Ad Q3. *TI-Ns seems to modify the tempos of some multimodal behaviours of students; this makes TI-Ns possibly similar to some new Representational Infrastructures used in nowadays technological society, e.g. the increasing habit of simultaneously surfing of youngest people through different technological devices for shorter period of times –the multitasking attitude- compared with the old way of operating in sequence for longer periods of time* (see Baricco, 2006; Tapscott & Willliams, 2007)

Generally speaking, using TI-Ns in the classrooms requires a strong effort from the students, but the effort is lived positively by most students because their impression is that TI-Ns can support their learning processes more than other software (see the answers to the questionnaires). Moreover the software seems to be accepted by students, because of its multitasking aspects, which are consonant with their technological habits outside the school.

2.1 The triadic structure of instrumented actions within TI-Nspire and the related tempos

To properly describe TI-Nspire instrumented actions we refine the usual dyadic structure epistemic-pragmatic of the instrumental approach. We introduce a fresh modality that we have called quasi-empirical and distinguish it from stricter epistemic ones. To give an idea of what is meant by quasi-empirical, I discuss sketchily a typical TI-Ns situation and contrast it with a Cabri-situation.

A simple problem, originated by the PISA test is the following:
Problem 1. *The students A and B attend the same school, which is 3 Km far from A's home and 6 Km far from B's home. What are the possible distances between the two houses?*
A possible solution with TI-Ns is illustrated in Fig. 3. You draw two circles, whose centre is the school: they represent the possible positions of the two houses with respect to the school. Then you create two points, say a and b, moving on each circle, construct the segment ab and measure it. Successively you create a sequence of the natural number in column A of the spreadsheet (*fig. 21a*) and through two animations (in one you move a and in the other you move b) you collect the corresponding lenghts of ab (columns B, and C in the spreadshhet of *fig. 21a*). In the end you make the scattered plot A Vs/ B and A Vs/ C (*fig. 21b*). Then you draw your considerations about the possible distances of A's and B's houses, considering the properties of the obtained scattered graph.

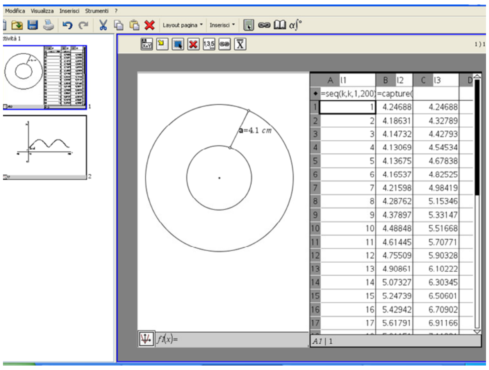


fig. 21a

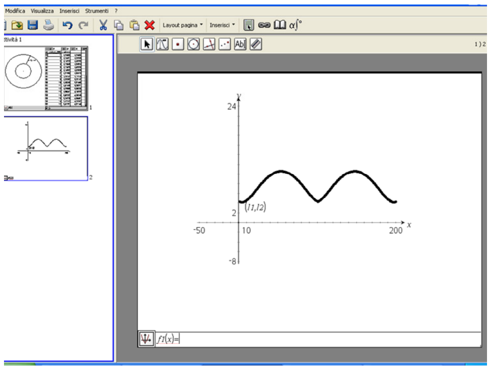


fig. 21b

Consider now the following problem to solve with Cabri (but it could be given also within TI-Ns):
Problem 2. *Let ABCD be a quadrangle. Consider the perpendicular bisectors of its sides and their intersection points A', B', C', D' of pairwise consecutive bisectors. Drag ABCD, considering its different configurations: what happens to the quadrangle A'B'C'D'? What kind of figure does it become?* (*fig.s 22a, 22b*)
Many students, who are able to use Cabri, after some initial explorations, where they drag almost casually some of the suggested points (we call this *wandering dragging*), almost by case observe that with particular positions, the four points A', B', C', D' coincide. So they try to carefully drag the vertices of the quadrilateral so to keep together the four points A', B', C', D' (we call this *dummy locus dragging*).
After some different explorations they realise that the coincidence situations happens when the four vertices of the quadrilateral have the same distance from the common point A', B', C', D'. Namely when the quadrilateral is cyclic. (For a detailed analysis of this and similar problems see: Arzarello, 2000 and Arzarello et al., 2002).

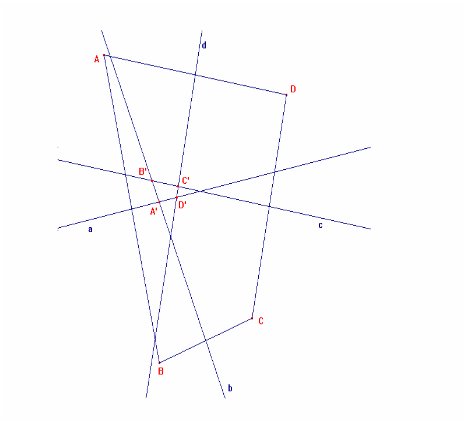


fig. 22a

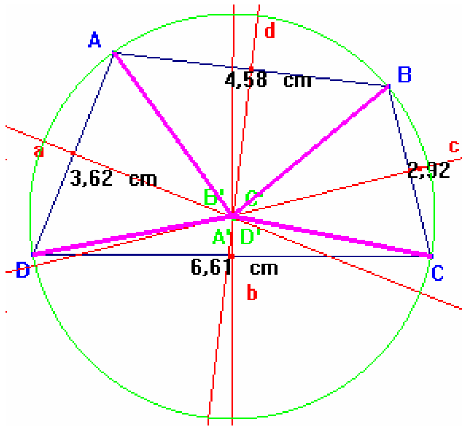


fig. 22b

The two examples are different for many reasons (typology of problem, different difficulties, etc.) but illustrate very well two different approaches: one is possible both in Cabri and in TI-Ns, the other is typical of TI-Ns but is almost impossible in Cabri for pragmatics reasons.
Let us consider the role of the variable points and the ways they are manipulated in the two cases. For both the Instrumental approach would speak of epistemic instrumented actions. But considering carefully the nature of the instrumented actions in the two environments one finds interesting differences.
In the example of TI-Ns (which is an emblematic example within this environment), TI-Ns allows a collection of data very similar to those accomplished in empirical sciences. The involved variables are picked out; then through the sequence A one gets a device to reckon the time in the animation in a conventional way: namely the variable time is made explicit[6]. Of course this subtle point is not so explicit for students: it is a practice induced by the instrumented actions of TI-Ns; students find natural to do so, since it works. It is interesting to observe that the scattered plot combines the time variable A Vs/ the length variable B or C. This practice is almost impossible or at least not simple in Cabri. The reason is in the possibility for TI-Ns of making the time variable explicit within mathematics itself. That is, given a mathematical problem (like Problem 1) one can do an experiment very similar to those made in empirical sciences: one picks up the (supposed) variables that are important for the problem; makes a concrete experiment that involves such variables; then the mutual relationships among them are studied (using the scattered plot) and a mathematical model is conjectured and tested (e.g. through new experiments). In the end possibly the reasons why such a model is got are investigated, namely a proof of a mathematical sentence is produced. All this happen in a very precise methodology (picking up of variables, designing the experiment, collecting data, producing the mathematical model, proving), which is made palpable by different canonical functions of TI-Ns (respectively, giving a name to the variables, animation or dragging, data capture, either scattered plot and its adjustements, or regressions and scattered plot): only the last one, proving, is not supported in a direct way by a precise function of the software. In Cabri there are two main differences with respect to this situation: first, it is not possible to explicit the variable time through the "Newton trick"; second, it is not possible to collect data like in TI-Ns: as far as I know, the only way is through trace and geometrical locus, namely by-passing the numerical aspects. The two differences are crucial. In fact in Cabri it is possible to make experiments, but without "the time of the environment" (that is it is possible to have time as a variable that refers to some external event, for example for studying the graph of y=t^2, where t is time in an abstract

[6] This procedure is very similar to the way Newton introduced his idea of scientific time in science, distingushing it from the fuzzy idea of time, about which hundreds of philosophers had (and would have) speculated. For this reason I call this procedure the "Newton trick" (see Arzarello, in print).

sense, but that does not refer to the "time" of what is happening in the Cabri-world). Some consequences of the second difference, which consists in systematically bypassing the numerical aspects (that is the spreadsheet), will be discussed in §2.2. Here I underline that this bypassing destroys the perfect correspondence between the empirical methodology stressed above and some precise functions of TI-Ns. In this sense TI-Ns introduces a new dimension in mathematics, which I call *quasi-empirical*. Such a methodology does not only consists in the possibility of making explorations: this happens in many software, specifically in Cabri through the different types of dragging (Arzarello et al., 2002), in Derive through the use of moving cursors. In TI-Ns it consists in a precise protocol that students learn to use and appears very similar to the way external data concerning certain quantities are got through the use of probes connected to a computer. In our case the probe are the measures in the "internal experiment" made in the TI-Ns mathematical world and collected from the G&G sheet to the Spreadsheet. From the one side, the methodology is empirical, but from the other side it concerns mathematical objects and not physical quantities. Hence the name quasi-empirical for this. The terminology is from Lakatos (1976): it is beyond the aims of this report to discuss the reason of this terminology (which has a precise didactical reason). Hence quasi-empirical instrumented actions are typical of TI-Ns.
Quasi-empirical actions have an epistemic nature, hence are not pragmatic; but they are specific of TI-Ns; hence we distinguish them from other epistemic actions that TI-Ns share with other software, e.g. exploring a situation with dragging or using a cursor for varying the parameters of an animation. An argument for this distinction is that in TI-Ns we find all the epistemic actions produced within the other main didactical environment, but the converse is not true.

There are also strong similarities between instrumented actions produced in TI-Ns and Cabri environments: they are revealed analysing the students' actions, productions and communications according to the multimodal paradigm. Namely, when working with Cabri as well as with TI-Ns in small groups, students in both cases use a variety of semiotic resources and not only the ones specific of software: gestures, drawing, speech. In this sense TI-Ns results similar to other DGS. However, also in this case there is an important difference, whose cognitive and didactic meaning will be discussed below. Namely, while in Cabri when students start writing their report, there is a strong break with the use of the instrument, in the sense that the exploration stops and a new phase starts where writing is prevailing, in TI-Ns this cut is less evident: students generally write a first part of the report, then come back to the other sheets of the instrument, make new actions and productions, and go back to the Notes. This probably happens because the Notes sheet is itself a part of the environment: the reporting phase is within the same environment as the experimenting one.

The presence of quasi-empirical instrumented actions seems to have important consequences for learning. Typically, comparing the epistemic actions in DGS with the quasi-empirical ones within TI-Ns, we find that in the first cases there is a big bridge between the perceptual level at which such actions (e.g. *wandering* and *dummy locus* dragging in Cabri) are drawn and the theoretical level, which is necessary to develop proofs or to build up models of the studied phenomenon.
In the first case (exploring-conjecturing-proving theorems in geometry), the transition to the deductive level is generally marked by a cognitive shift from *ascending to descending* modalities[7], according to which the figures on the screen are looked at: usually the shift is marked through an

[7] The modality is ascending (from the environment to the subject) when the subject explores the situation, e.g. a graph on the screen, with an open mind and to see if the situation itself can show her/him something interesting; the situation is descending (from the subject to the environment) when the subject explores the situation with a conjecture in her/his mind. In the first case the instrumented actions have an explorative nature (to see if something happen), in the second case they have a checking nature (to see if the conjecture is corroborated or refuted). For more information, see Saada-Robert (1989), Arzarello (2000), Arzarello et al. (2002).

abduction, which also shows the transition from an inductive to a deductive approach. Generally speaking, the *tempos*, according to which such phenomena develop in time are long and require a high ingenuity by the pupils. The software is a support but the bridge to gap may be very large.
In the second case (building up models of phenomena), different variables are to be picked up for building a model of the studied phenomenon through functions: in this case, students are supposed to grasp which variables are functions or which in order to use the calculator function to implement the graph of such functions through trace and/or through the locus function.

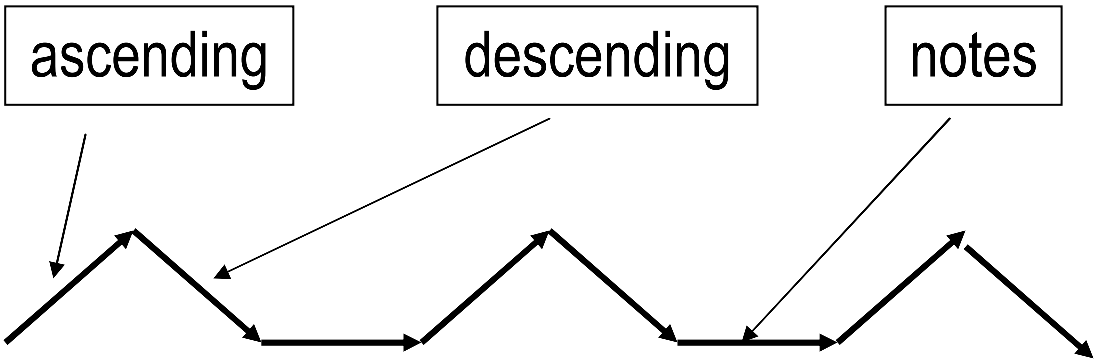


fig. 23a (TI-Ns tempos)

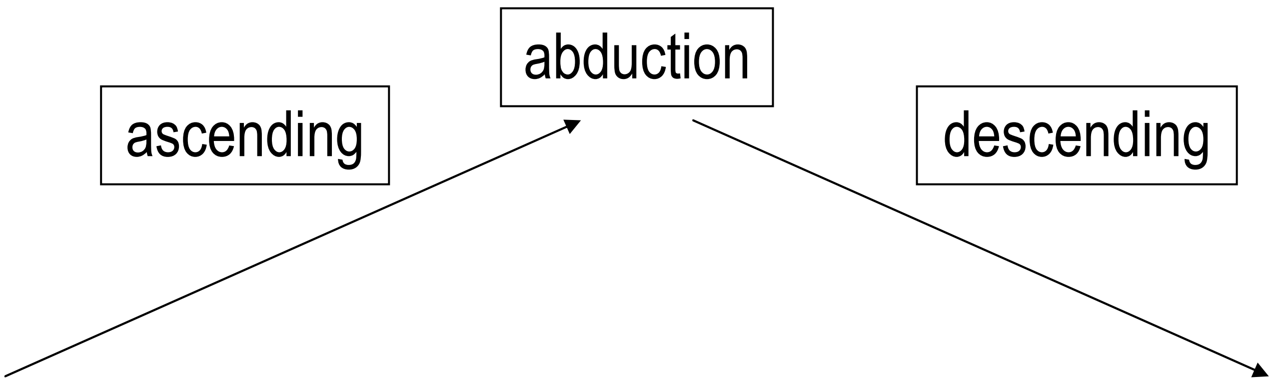


fig. 23b (Cabri tempos)

In the case of TI-Nspire, we have two different evolutions, both due to the quasi-empirical modalities of the instrumented actions drawn in this software. In the case of tempos, instead of a unique long episode with the transition from the ascending to the descending modality, we generally have many shorter episodes, each marked by the use of a different function of the software: first the data capture, then the scattered plots (which can be done without yet knowing which variable is function of which), then the interpretation of the graph. Each action corresponds to very short alternation of ascending-descending modalities.
Moreover, because of the didactic contract, these shorter phases generally contain an epistemic part, where the students write in the Notes what they are doing, their conjectures etc, without breaking the quasi-empirical process they are developing (a further big difference with Cabri). Instead of a unique long instrumented epistemic action, we have many short and different instrumented quasi-empirical actions (see *fig.s 23a, 23b*).
This breaking into smaller units makes easier for students to manage their practices in the new environment (once enough pragmatic actions have been developed in order to acquire a suitable knowledge of the instrument): the cognitive load is smaller. Moreover their smaller actions are instrumented through some precise specific functions of the software. This makes it possible to institutionalise more easily the corresponding practices and less ingenuity is needed.

Of course, there are two issues that must be developed in the classroom before this can happen: first TI-Ns requires more pragmatic actions to be learned at a level sufficient to use it as a suitable tool to solve problems with the support of the teacher. In the two classrooms of our teaching experiment 12 hours have been necessary. In this period the major part of student actions have been pragmatic, while only a minor percentage has been classified as quasi-empirical or epistemic. After the apprenticeship period the students show a diminishing of the pragmatic actions and a sharp increasing of their epistemic and quasi-empirical ones. We have also analysed the protocols of pupils that used Cabri in solving problems and that knew the software very well: the percentage of time that they spend in producing epistemic actions is comparable with the one that students using TI-Ns spend in producing quasi-empirical + epistemic actions (with a proportion of 2 to 1 in favour of quasi-empirical actions with respect to other epistemic ones). In both environments the epistemic and the quasi-empirical actions have a strong perceptuo-motor nature: the main difference between the two consists in the different alternating shorter ascending-descending phases in TI-Ns with respect with longer one way ascending and descending phase in Cabri.
The impression is that such a difference stresses an attitude that tunes more with nowadays youth, who like better a multitask way of operating with ICT, doing many things together within many modalities but for shorter periods of time than concentrating for a longer time within a unique modality (see the surfing way of life, described in Baricco, 2006). The quasi-empirical actions with all their multimodal aspects and particularly their tempos feature some fresh praxeologies that enter the classroom and are in positive resonance with practices within fresh representational infrastructures (for this notion see Kaput & Noss, 2002) that are more and more diffuse in our society. The APCs built up by students using TI-Ns uses such modalities that come from the world outside the school (external practices). This may enhance the cognitive capabilities of our students in learning mathematics insofar it supports performances and modalities that do not require *ad hoc* learning.
This issue is particularly evident in the class of Pierangela, where a part of the students, in particular those not so comfortable with the systematic traditional way of working in the classroom, have been able to use their extra-school 'multitasking' competencies and their usual ways of solving everyday problems by trials and errors in approaching mathematical problems. Such competencies have been supported and improved by the multi representational environments of TI-Nspire and have given good results. It is a pity that in our experimentation we have not allowed to use the device that permits to connect the different handhelds in the classroom: the new communication environment, which would have further fostered the resonance with such external practices (see below 3.2). Some partial results in this sense have been reached creating a forum in the two classes.

2.2 Towards a meaningful use of symbols

The second specific feature, which distinguishes TI-Ns from other software, is the instrumented actions that students develop using the symbolic spreadsheet of TI-Ns. This is a strong didactical innovation.
I will illustrate this issue through an example from the 9-th grade students, who were studying functions through their tables of differences. They had already learnt that for first degree functions the first differences are constant. They were asked to make conjectures on what functions have the first differences that change linearly.
Their conjecture was that quadratic functions have this property and arranged a spreadsheet like in Fig. 6a, where they utilised:

- columns A, B, C, D to indicate respectively the values of the variable *x*, of the function *f(x)* (in B_i there is the value of $f(A_i)$) and of its related first and second differences (namely in C_i there is the value $f(A_{i+1})$-$f(A_i)$ and D_j there is the value C_{j+1}- C_j);
- variable numbers in cells E2, F2, …,I2 to indicate respectively: the values x_0 (the first value for the variable x to put in A2); *a*, *b*, *c* for the coefficients of the second degree function

$ax^2+bx + c$; the step h of which the variable in column A is incremented each time for passing to A_i to A_{i+1}.

Modifying the values of E2, F2, …,I2 the students can easily do their explorations. This is a learnt practice, that gradually becomes a praxeology in the classroom, because of the interventions of the teacher, who stresses its value as an instrumented (epistemic) action, which supports explorations in the numerical environment.

It is interesting to observe that such a praxeology reveals its didactical power if analysed through a semiotic lens. Using the terminology in Duval (2006), this instrumented action supports a systematic *treatment* of numbers, scaffolded according to the formula of the second-degree function. This types of treatments, according to the frame of Duval, is one of the roots for developing algebraic thinking in students. Hence the instrumented actions of this type seem to be promising in the learning of algebra. In this TI-ns is not different from usual spreadsheets and share with them such potentialities with respect to the learning of the algebraic language.

	A	B	C	D	E	F	G	H	I
♦									
1	x	f(x)	df(x)	ddf(x)	x0	a	b	c	h
2	0	3	-1	-4	0	-2	1	3	1
3	1	2	-5	-4					
4	2	-3	-9	-4					
5	3	-12	-13	-4					
6	4	-25	-17	-4					
7	5	-42	-21	-4					
8	6	-63	-25	-4					
9	7	-88	-29	-4					
10	8	-117	-33	-4					
11	9	-150	-37	-4					
12	10	-187	-41	-4					
13	11	-228	-45	-4					
14	12	-273	-49	-4					
15	13	-322	-53	-4					
16	14	-375	-57						
17	15	-432							

fig. 24a

	A	B	C	D	E	F	G	H
♦								
1	x	f(x)	df(x)	ddf(x)	x0	a	b	c	h
2	x0	a*x0^2+b*x0…	a*(h^2+2*h*x0)+…	2*a*h^2	x0	a	b	c	h
3	h+x0	a*(h+x0)^2+…	a*(3*h^2+2*h*x0…	2*a*h^2					
4	2*h+x0	a*(2*h+x0)^2…	a*(5*h^2+2*h*x0…	2*a*h^2					
5	3*h+x0	a*(3*h+x0)^2…	a*(7*h^2+2*h*x0…	2*a*h^2					
6	4*h+x0	a*(4*h+x0)^2…	a*(9*h^2+2*h*x0…	2*a*h^2					
7	5*h+x0	a*(5*h+x0)^2…	a*(11*h^2+2*h*x…	2*a*h^2					
8	6*h+x0	a*(6*h+x0)^2…	a*(13*h^2+2*h*x…	2*a*h^2					
9	7*h+x0	a*(7*h+x0)^2…	a*(15*h^2+2*h*x…	2*a*h^2					
10	8*h+x0	a*(8*h+x0)^2…	a*(17*h^2+2*h*x…	2*a*h^2					
11	9*h+x0	a*(9*h+x0)^2…	a*(19*h^2+2*h*x…	2*a*h^2					
12	10*h…	a*(10*h+x0)^…	a*(21*h^2+2*h*x…	2*a*h^2					
13	11*h…	a*(11*h+x0)^…	a*(23*h^2+2*h*x…	2*a*h^2					
14	12*h…	a*(12*h+x0)^…	a*(25*h^2+2*h*x…	2*a*h^2					
15	13*h…	a*(13*h+x0)^…	a*(27*h^2+2*h*x…	2*a*h^2					
16	14*h…	a*(14*h+x0)^…	a*(29*h^2+2*h*x…						
17	15*h…	a*(15*h+x0)^…							

fig. 24b

But this is only the first half of the story. There is a more interesting second part, which happens because of the symbolic calculations that TI-Ns can support with its spreadsheet.

In fact students realise that:

- if they change only the value of c, only column B changes, while the columns C and D of the first and second differences do not change; hence they argue that the fact that and the way how a function increases/decreases does not depend from the coefficient c;
- if they change the coefficient b, then columns B and C change but column D does not; many students conjecture that the coefficient b determines if a function increases or decreses but not its concavity;
- if they change the coefficient a, then columns B, C and D change; hence it is the coefficient a to be responsible of the concavity of the function.

A difficult point here is to understand why such relationships hold. The tables of numbers do not suggest anything. It is the symbolic power of the spreadsheet to be useful in this case. The epistemic instrumented action in this case is very interesting and consists in substituting letters to the numbers: see *fig. 24b*. This practice has been suggested by the teacher in most cases but a couple of students has done it autonomously. The spreadsheets shows clearly in this case that the value of the second difference is $2ah^2$. In this case the letters condense the symbolic meaning of the explorations developed before in the numerical environment.

The teacher has stressed this power of the symbolic spreadsheet in the next lecture and again a fresh praxeology has entered the classroom.

In this sense TI-Ns allows an anticipated exposition of students to the symbolic aspects of the mathematical language supporting suitable instrumented actions, which are particularly apt to trigger the symbolic function of the algebraic language.

3.The handheld technology

Let us now come to the question MQ2, concerning the effects of using the handheld instead of (or with) the PC TI-Nspire: *are there differences between the learning processes supported by the two different environments of TI-Ns?*

I recall here the first three questions derived from the elaboration of the question MQ1 and the answers we have found:

Questions:

Q1 *How do praxeologies change in the classroom because of the introduction of TI-Ns? How the fresh praxeologies (if any) help(or block) students in learning mathematics?*

Q2 *What is the specificity of instrumental actions in TI-Ns environment*? *What instrumented actions in TI-Ns help (or block) students learning processes?*

Q3. *At what extent does TI-Ns modify the usual multimodal behaviours of students?*

Answers:

Ad Q1. *New praxeologies are introduced in the classroom because of fresh specific instrumented actions supported by TI-Ns. Some of them have a positive consequence on learning processes of the students and on their attitudes towards mathematics.*

Ad Q2. *The major new entries in the instrumental actions supported by TI-Ns concern the specificity of the transition to the theoretical side of mathematics, to its modeling and to a meaningful introduction to the use of symbols.*

Ad Q3. *TI-Ns modify the tempos of some multimodal behaviours of students; this makes TI-Ns possibly similar to some new Representational Infrastructures used in nowadays technological society, e.g. the increasing habit of simultaneously surfing of youngest people through different technological devices for shorter period of times –the multitasking attitude- compared with the old way of operating in sequence for longer periods of time* (see Baricco, 2006; Tapscott & Willliams, 2007)

To translate the question MQ2 in a more scientific language I start from a theoretical frame, elaborated by Borba and Villareal (2005), namely the so called *Humans-with-media* (HwM from now on) approach. It allows to suitably focus the phenomena of interactions that happen when the students use tools like the TI-Ns handheld, through which their products can be easily be shared in the classroom, and consequently the social construction of knowledge becomes an important component in learning processes. Successively I shall make a comparison with some social phenomena that happen in the classroom using specific technological tools (namely TI-navigator) and will categorize them in two main complementary branches (between Vs within phenomena). Comparing the two environments (TI-Ns vs TI-navigator) I shall illustrate that they embody each one of the two dimensions but not the other, hence they are complementary but incomplete: the handheld environment could be a more complete tool, provided it could be supplied of some facilities, that were not present in the technology we used in the experimentation, and modulo some difficulties that students find in managing the keyboard.

3.1 Humans-with-media.

Humans-with-media is a theoretical approach that takes into account both the subjects and the tools involved in a mathematical activity (Borba and Villarreal 2005) and it is grounded on two ideas: first, the construction of knowledge is made in a social way, by subjects working together; second, the media involved are part of this construction, because they collaborate to reorganise thinking, with a different role than the one assumed by written or oral language. Borba and Villarreal introduce a point of view that contains and enlarges previous instrumental approaches that take into account only the relationships between a subject and a technology. They focus on the community of learners (small groups, as well as the whole class or bigger groups), along with the tools. This point of view overcomes the deep-rooted dichotomy between humans and technology. It suggests that learning is a process of interaction among humans as a group, including tools, which are seen as ‘actors’ in a collective thinking, in the sense that they are carriers of a historical-cultural heritage and mediate the construction of knowledge. So media interact with humans, in the double sense that technologies transform and modify humans’ reasoning, as well as humans are continuously transforming technologies according to their purposes. This framework is particularly suitable for describing the situation that we have in a classroom while using the handheld, provided the results of the students are systematically shared and discussed in the classroom (like it now can happen with the TI Connect-to-Class Software). In our experiments we got such a sharing through more rudimentary methods, which made our results less neat and probing.

3.2 Between- Vs Within- multi-representations.

Let me consider such a general frame for the specificity of the TI-Ns handheld. In the previous chapters I have illustrated how the *multi-representations* in the TI-environments can trigger and support students processes of mathematics learning (through a suitable didactical design), focusing their strong (and feeble) features. I have also pictured how the specific *mediations* of the TI-instruments can support students’ learning processes. Now I wish to consider the features of interactions (students-students, teacher-students) that can happen in the classroom when using TI-Ns software. In particular I will consider the role of the teacher and what new opportunities the handheld technology offers to her/him in designing and managing mathematical activities. In developing my line of analysis the comparison between the two environments TI-Nspire Vs. TI-Navigator will be very useful.

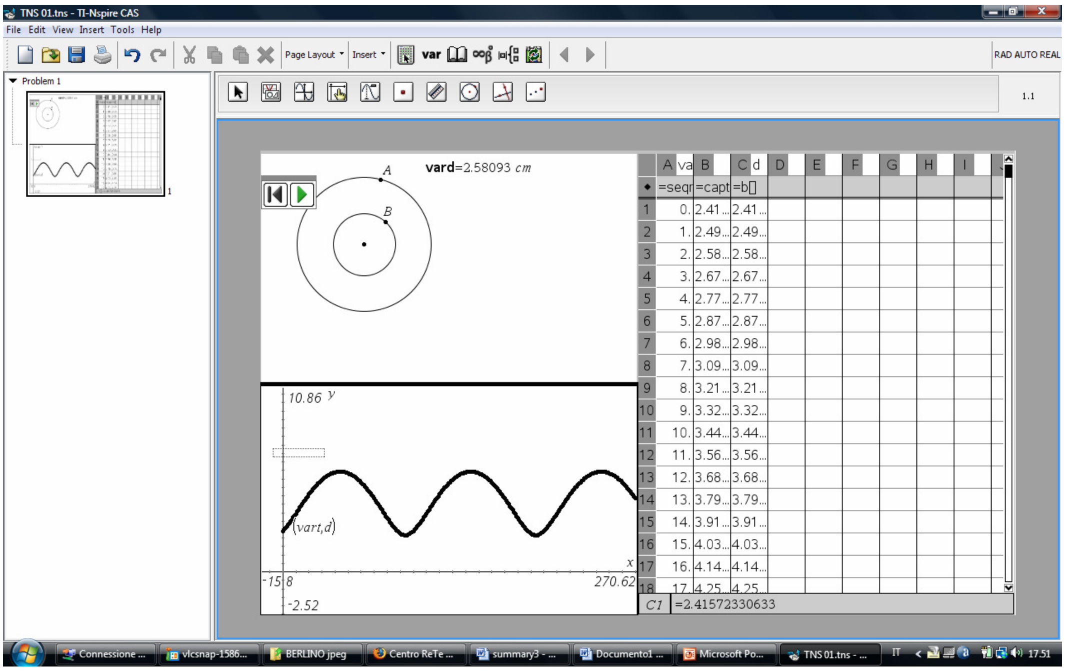


fig. 25

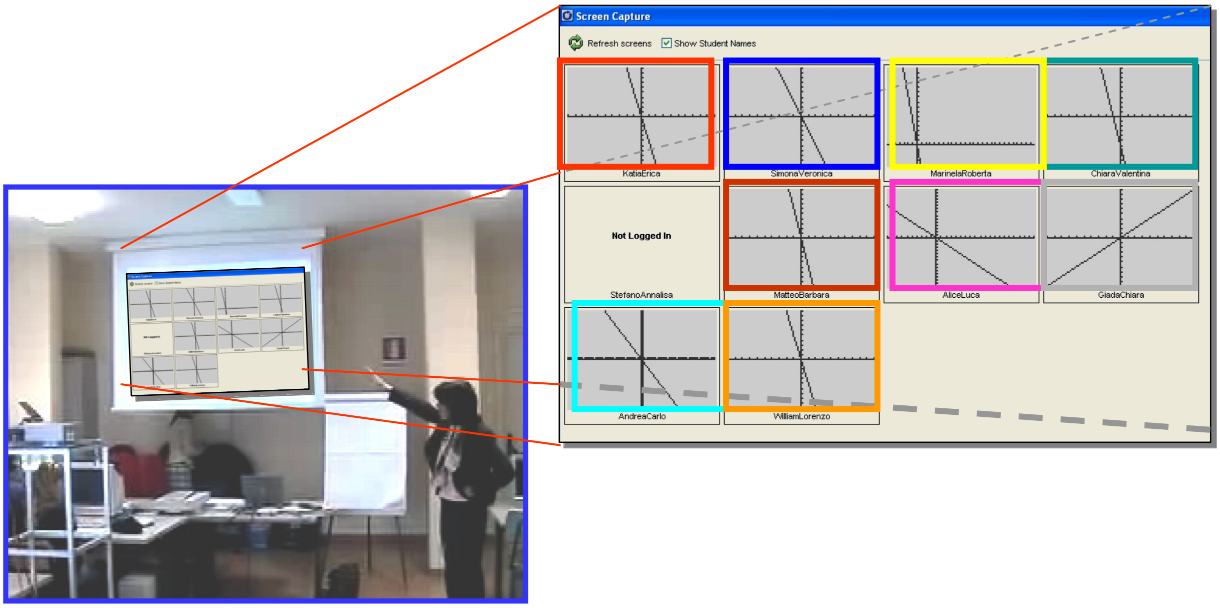


fig. 26

Fig.s 25, 26 illustrate two main specific features of TI-Ns and of TI-navigator, which consist in two dual ways of implementing multi-representations; in fact we can have:

- different representations in different registers produced by the same subject (TI-Ns, Fig. X1)
- different shared representations produced by different subjects within the same register (TI-Navigator, Fig. X2)

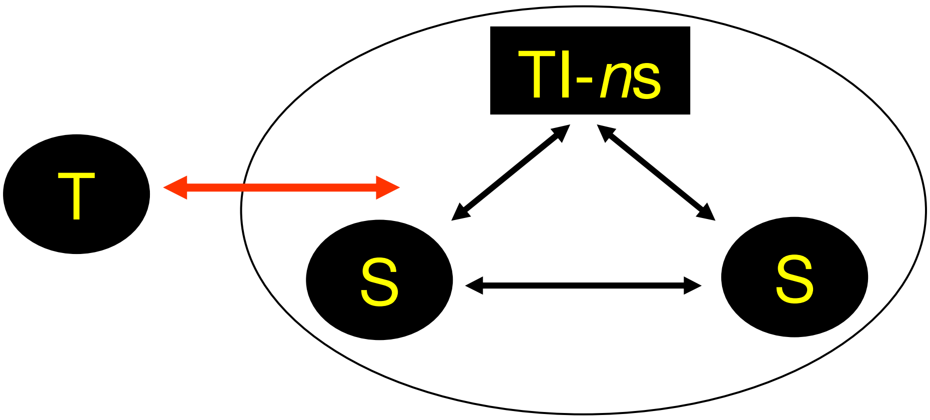


fig. 27

Fig.s 27, 28 illustrate the different interactive situation that we have in the two environments. Correspondently we have two different functions of representations in the classroom interactions:

Within-multi-representations (WMR): they concern the interactions of the students with the multi-representations supported by the software itself (e.g. TI-Nspire): typically, the geometrical, the algebraic and the spreadsheet treatment of the same problem produced by a single student or by a small group of students using the same device

Between-multi-representations (BMR): they concern principally the interactions that the instrument (e.g. TI-Navigator) triggers and supports among the students in the classroom, because of the simultaneous access on the shared screen to the solutions produced by different students for the same task.

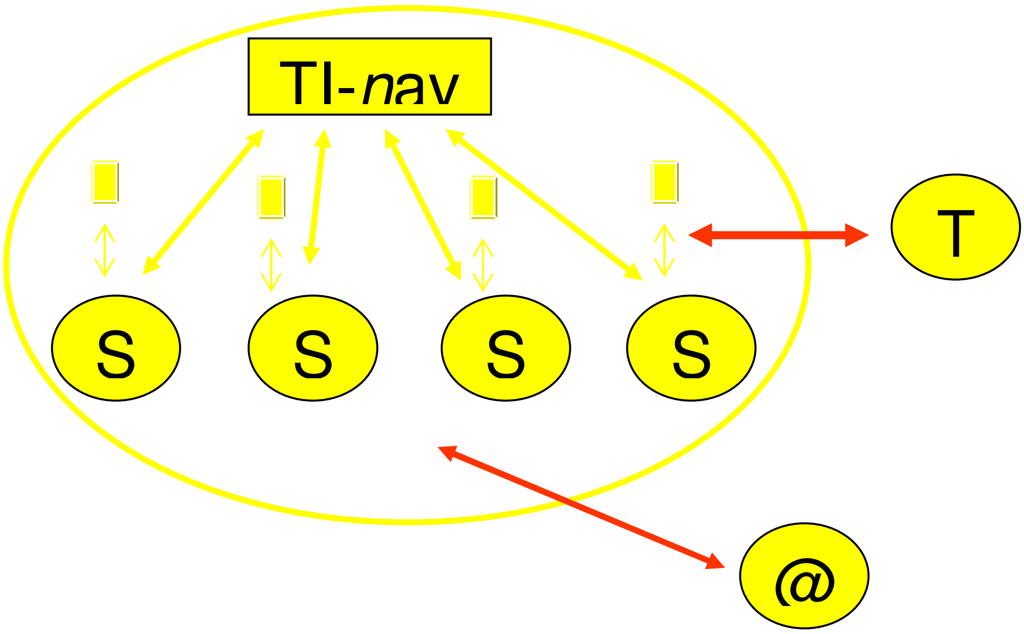


fig. 28

While the WMR are very well triggered and supported by the TI-Ns environment, it is not so for the BMR. The handheld integrated with the Connect to Class software offers both the functions (*fig. 29*).

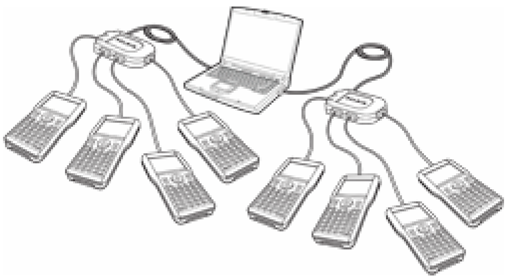


fig. 29

It is clear that while the WMR can give reason of all the aspects of learning processes I illustrated in the first two chapter, the BMR functions are the right point to look at considering how the technological environment can trigger and support the interactive and social aspects of knowledge building in the classroom, namely those aspects particularly considered in their frame by Borba and Villareal. We tried to overcome this missing aspect of the TI-Ns technology we had during the experiment, using the opportunities that the internet connections and the use of a shared platform in our laboratory could offer. Table 3 illustrates the different features of the two interactive modalities.

	Registers/ Subjects	Interactions
Within m-repres (TI-*n*spire)	Many/One	One-Few subjects Vs One Instrument
Between m-repres. (TI-Navig)	One/Many	Between all subjects each with one Instrument

Table 3

In general the WMR supports more the so called “jeu des cadres” (Douady, ZZ), e.g. from the geometrical to the algebraic, as illustrated above, and consequently the conversions between registers, e.g from the numerical environment of the spreadsheet to the geometrical on of Graphs and Geometry. From the other side, the BMR supports more the so called "mathematical discussion", which grows up in the classroom when different solutions are compared or contrasted. Moreover the different multi-representations support different cognitive processes. WMR fosters more "surfing" modalities among the different environments and students are looking more for connections between them. BMR fosters more "chatting" modalities among the students and they are more looking for sharing, contrasting, comparing their different points of view. Both modalities are important and very useful in learning processes and illustrate two different "dual" modalities, according to which humans and media co-exist and intertwine in the Borba and Villareal frame. When the teachers disposes of both modalities it is easier for her/him "making the point" and "institutionalising" the achieved new knowledge in the classroom.

3.3 An example in positive: between- and within- multi-representations at work

Reconsidering our answer ad Q3, we can refine the answer and say that BMR and WMR of TI-environments amplify the multimodality of the semiotic resources that can be used by the subjects. It is interesting to observe how the integrated use of both BMR and WMR in the classroom can improve also the use of the paper and pencil environment, integrated with the technological tools. Learning through an integrated use of different resources can be so improved. Each one amplifies the use of the other, as pointed out in Kieran and Drijvers (2006).

To illustrate this, I shall dedicate some space to comment what happened in the classroom of Domingo (10^{th} grade) during a problem solving solution with the TI-Ns handheld. The example illustrates how the combined use of technologies that support BMR and WMR functions with paper and pencil environments can offer the teacher first the opportunity of focusing subtle but important mathematical problems not so easily accessible in only one environment, and second the tools for a positive mediation with respect to the consequent difficulties met by the students. The "combined environment" can be thought as a tool that triggers problem posing and supports problem solving activities, provided the teacher suitably designs her/his interventions. The example we discuss here is emblematic of similar cases we met in the teaching experiments with TI-Ns. I recall that the curriculum followed by Domingo is "function-based" (Chazan and Yerushalmy, 2003).

The combined approach philosophy ensues from the following observations. From the one side, the students, who solve problems within technological environments, often develop practices that are significantly different from those induced by paper and pencil environments and this may offer fresh didactical opportunities:

> the curriculum with technology…changes the order and the intensity in which students meet key concepts. This change in order allows students to solve some kinds of problems that students typically might find difficult; it also either restructures points of transition between views or introduces new points of transition. (Yerushalmy, 2004, p.3).

From the other side, sometimes they "naturally" use a mixed approach, where paper and pencil environment survives beside the technological one and we have never discouraged such an integrated use of their different resources. In such cases it can be useful to exploit the didactical positive interactions of the two, suitably designing their combined use. We have observed that this methodology can be particularly useful in approaching some delicate mathematical problems, where remaining within only one environment (technological or not) may not be so productive. We shall illustrate this point showing how students choose the independent Vs dependent variables for modelling sequences of geometrical figures defined by recursive rules.

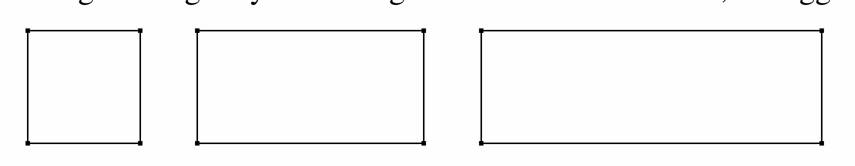
In the activity we analyse, the students, first grouped in pairs, than sharing their sound solutions, must solve a problem taken from Hershkovitz & Kieran (2001), according to the following task sheet:

Task. "Listen carefully to the reading of the problem by the teacher. For 10 minutes think individually to the problem: do not use paper and pencil or TI-nspire. Produce conjectures about the change of the rectangles areas. In the successive 10 minutes discuss your conjecture with your mate; use paper and pencil only; share possible strategies to approach the problem (for validating or exploring) within TI-nspire. In the successive 60 minutes you can use TI-nspire to verify your conjectures, to explore the problem and eventually to solve it. In the next lecture you will compare and discuss your solution with those of the other groups."

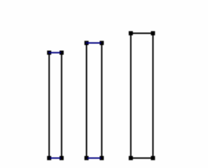
In fact after the group work, the whole class discusses and contrasts the found solutions and the teacher institutionalise the right knowledge. During the discussion the students see all the productions they have done, in a TI-navigator like environment.

Problem. "Consider the following three sequences a), b), c) of rectangles:

a) The height is constant (1 cm); the base of the first rectangle is 1 cm, while the successive rectangles are got by increasing the base 1 cm each time, as suggested by the following figures:



b) The first rectangle has height of 1 cm and base 0.1 cm; the successive ones are got increasing of 0.1 cm both the base and the height each time, as suggested by the following figures:



c) the first rectangle is a square with the side of 0.01 cm; the successive rectangles have the height always of 0.01 cm, while their bases are got each time doubling the base of the previous rectangle, as suggested by the following figures:

What can you say about the type of growing of the rectangles area in each sequence?
Justify your answer."

h	B
1	0.1
1.1	0.2
1.2	0.3
1.3	0.4

Table 4

h	B
1.9	1
2	1.1
2.1	1.2

Table 5

All the pairs have produced a final document within TI-*n*spire, which has been shared and discussed in the classroom. I shall comment the strategies elaborated by L and S (two good level achievers in mathematics) to solve the three questions. I shall analyse what happened only in the last two phases of their work (with paper and pencil and with TI-*n*spire) and shall summarise the results of the final discussion.

In phase 2, L and S do not hesitate to agree that the area in a) changes linearly. The study of the sequence b) is not so immediate. L and S build a 2 columns table, where they write the first values of the height and of the base (Table 4). L observes that the areas seem to "grow more and more" (it is the shared expression to indicate a function that increases with the concavity upwards). L wonders if this type of growing can concern all the data and not only the few considered in the table. His conjecture is that it is so provided the base does not exceed 1.

Hence he builds a second table (Table 5), which starts with the value 1 in the second column. This method is a typical strategy within paper and pencil environment; using the spreadsheet of TI-*n*spire the strategy would have been different, since students could have easily considered a lot of values and studied them with the first and second differences. At this point L generalises his conjecture saying: "It seems that it grows more and more…even because if one enlarges…it must grow more

and more…two sides are always growing…hence it must grow"; and with the pencil traces in the air the "drawing" of an increasing curve with concavity upwards.
Then they pass to the sequence c). Also in this case the two students produce a table like above. At this point the teacher interacts with them and asks them what kind of growing they expect. S makes a gesture, which in their previous discussion had been used to indicate the doubling of the base. L says explicitly: "exponential…there are always powers of 2". Then L calculates some first differences, observes that they reproduce the same values of the function and this confirms his conjecture of an exponential growing. Even if with some perplexity S accepts. Hence the students are ready to pass to the software already with many given answers. One could so expect that in TI-*n*spire they find the confirmation of their (right) conjectures. This regularly happens with the sequence a): the graphic and numeric information they get from the software are coherent each other and confirm their conjecture of a linear growing. More interesting their work for the sequence b). Once they have done the work with the spreadsheet of TI-*n*spire they wish to produce a graphic and must decide what is the independent and what the dependent variable. The second choice is obvious: it is the area. But what about the independent variable? They have some uncertainty:
L: With respect to the variation of what? Of the base?
S: Hmm…
L: Yes, L3 [he refers to the name of the variable in the spreadsheet]
S: However, it is not only the change of the base ...
L: Both are changing…both are changing… with respect to the variation of what otherwise?
S: Yes but both are changing…
After a while, the teacher recalls them that when its second differences are constant the function is quadratic and then asks them: "that this is a second degree growing, could we have foreseen it?". The students remain silent for a while; then there is the following interaction between L and T (the teacher):
L: Hence they both increase [namely height and base]
T: Before you told me that when you have thought individually you thought to the fact that to find the area you multiply the base times the height. Isn't it? You have thought to this formula…
L: Yes, hence the area could be…Then we multiply the starting number…area-one equals b times a number, b times c. The second area equals (b+1) times (c+1), hence…
L gets lost with these computations: the symbols he uses are not so good to clarify why the sequence is quadratic.
T: Of what type is the change of the base?
L: Linear as that of the height.
T: If both the base and the height grow linearly, what happens to the area?
L: The area will grow…two things that grow linearly and are multiplied…ah yes x times x!
Hence they decide that the independent variable may be indifferently either the base or the height and draw the consequent graph with TI-*n*spire: a quadratic function of the area Vs the base.
The work for the sequence c) is very interesting. The students wait for an exponential graph, but when they draw the graph area Vs base a linear function appears! The graph is so unexpected that L suggests not to consider it and eliminates it from the screen of TI-*n*spire. It is the teacher to oblige them to reconsider what has happened.
T: What about the third graph?
L: Ehmm…we do not understand, it seems that it is a linear function […]
T: What were you waiting for?
L: More and more…[i.e. a growing function with the concavity upwards]
T: The area is growing […] why?
L: Because as a base…Because we have put…also the base is changing…
it changes with the *same step*.
T: Hence it is correct, isn't?
S & L: Yes, yes, yes […]

T: Be careful! We were waiting an exponential function. Namely the area were increasing exponentially, but with respect to what?
L: Of an *x* that *went on regularly…*
T: Well, what is this *x* that changes regularly? […]
L: With a *constant increment*
T: Yes but in what manner…when you have said that the area grows exponentially […] with respect to what you have thought it was increasing?...Not with respect to the base. In fact if the base grows up exponentially it is clear that the area …if the base doubles, the area doubles with respect to what?
L: With respect to what? […]
T: The area of the first rectangle is[…]
L: 0.0001
T: The area of the second rectangle measures…
L: Ah, *with respect to the places*.
T: Good, with respect to the places! This problem does not appear in the preceding sequences: why?
L: Because all change with a *constant step…the base*

It is interesting to observe how the students arrive to the linearity of the graph in the dialogue (see italics) and their explanation in the notes: "…the area of the sequence grows exponentially. This appears very clear to us looking at the values of the first and second differences [of the base], which result the same as those of the area". Namely for them it is clear that linearity depends on the choice of the independent variable [the base], which in this case changes proportionally with respect to the areas. So clear that they do not feel the necessity of making it explicit in their notes.

Of course not all the groups arrive to make it explicit this delicate point. Here is an example from another pair of students (T is again the teacher).

T: What can you multiply by two?
A: The area.
T: What area?
A: The area of the two rectangles, …is always multiplied by two.
T: In which sense? Can you explain me?
A: Because … well, one side remains constant, then the height remains constant, while the basis doubles at every step, so also the area, consequently, doubles at every step.
T: Then to find the 184th area, how do you do?
O: You have always to find the previous one … Every time.

The difficulty in finding the right model for expressing the area of the third family as a function of the step is evident in this protocol, where the awareness of the recursive law emerges, but not more.

It is only later, during the class discussion in front of the projection of all the screens of the groups, that all the pairs realize that the function is "a power of 2". Seeing what all the groups have done and because of the clever teacher's orchestration of the discussion in the classroom almost all the students in the end grasp what is the real function in case c).

This example shows the efficacy of the integrated use of BMR and WMR in promoting students solution processes and the relevance of the role of the teacher.

Let me see something about the role of the teacher and come to the students. The teachers while working in the classroom where a software like TI-Ns is used must essentially make what we call "semiotic mediation". In fact, according to Vygotsky's conceptualization of Zone of Proximal Development (Vygotsky, 1978, p. 84), teaching consists in a process of enabling students' potential achievements. The teacher must provide the suitable pedagogical mediation for students' appropriation of scientific concepts (Schmittau, 2003). Within such an approach, some researchers (e.g. Bartolini & Mariotti, 2008) picture the teacher as a *semiotic mediator*, who promotes the evolution of signs in the classroom from the personal senses that the students give to them towards the scientific shared sense. The semiotic mediation of the teacher is crucial to support the students

towards a deep understanding of the functional relationships among the variables of our problem. As a consequence, they can make an aware choice of the independent variables and draw a graph that suitably represents the situation. But this can happen because the semiotic mediation of the teacher can benefit of the intertwined effect of BMR and WMR. More precisely, even if the three questions a), b), c) are essentially solved by some students in paper and pencil environments (e.g. it is so for the first pair of students), they are solved only at different levels of understanding. Students are pushed to enter more deeply into the relationships among the variables that model the different situations by the instrumented actions they produce with the software. In fact, they must choose a column of the spreadsheet as independent variable to validate with the software what they are waiting for: the task is obvious in case a); problematic in case b), very difficult in case c). We call this the *problem of the independent variable.* In case b) they acknowledge that the quadratic dependence results because of the increase given to both the height and to the base of the rectangle. The reflection about the structure of the area formula (suggested by the teacher) generally produces the understanding of the real nature of the quadratic law (L says: "The area will grow…two things that grow linearly and are multiplied…ah yes x times x!"). The semiotic mediation of the teacher is based on two ingredients: (i) the necessity of passing from the signs of the spreadsheet to those of the graph environment of TI-*n*spire, which requires to explicit the two variables of the graph; (ii) the reflection on the way the multiplicative area formula incorporates twice the linear increment of the sides (bilinearity of the area function). It is the combined effect of these two ingredients to support the cognitive processes of L (and of other students). The third case is more complex: none of the variables in the spreadsheet changes linearly with the "place". The place is a hidden variable that has supported all the previous thinking processes of the students in cases a) and b). When passing to the software, the student change the independent variable, without realising it. But while in case a) and b) the hidden variable could in some way be represented through the variables they had in the spreadsheet (case b already posed some difficulties), in case c) this is not any longer possible: it is now necessary to explicit the hidden place-variable, to see what they are waiting for. The problem could not have cropped out so "naturally" in the paper and pencil environment. Students' instrumented actions generate it in cases b) and c) through a WMR but it is the intervention of the teacher or the final discussion in the group to make most students aware of the problem through a BMR. Its solution is crucial for developing an algebraic thinking apt to sustain the formal machinery that is necessary for modelling mathematical situations. It requires to shift from the neutral reading of the relationships among the variables of a formula (e.g. Area = base $\times$ height) to a functional reading of the same formula (e.g. Area = linear function of the base, provided height is constant, as in a). The epistemological relevance of this shifting was already pointed out by J.L. Lagrange (1879, p.15): "Algebra…is the art of determining the unknowns through functions of the known quantities, or of the quantities that are considered as known". Its didactical relevance has been stressed by many researchers, e.g. see Bergsten (2003, p.8).
Comparing what happened in our classroom with the results in Hershkowitz & Kieran (2001), who used a different software, we find some analogies and some differences, which stress the power of TI-Ns. Our experience is more similar to what happened in their Israeli 9th grade classroom, where students "were first invited to suggest hypotheses without using the computerized tool, then to use it to check them" (*ibid.*, p. 99). In that case students could find the closed algebraic formulas for problem c), even if with some difficulties; successively they could draw the three graphs using the graphic calculator. We must observe that the focus of the problem in that experience concerned more the comparison among the relative growth of the rectangles, while in our case the attention is more on the choice of the independent Vs dependent variables. During the discussion with the teacher, the Israeli students were able to match "together representatives from different representations: the algebraic, the numerical, the graphic, and the phenomenon itself" and "the evidence provided by the different representations of the software was accepted even if, for some students, it was unexpected" (*ibid.*, p. 100). In our case the students concentrated more on the finite difference techniques and got a meaningful model of the situation; however their successive

instrumented actions with the TI-Ns spreadsheet disorientated them because of some unexpected answers, particularly in case c). In our case the software acted also as a source of problems and it has been necessary a further strong mediation of the teacher. In fact, the independent variable problem is a subtle question that has been grasped by the students because of the instrumented actions fostered by the use of the spreadsheet and of the semiotic mediation of the teacher. The two have produced a meaningful reflection on this issue and avoided that "computerized tools reduce students' need for high level algebraic activity" (*ibid.,* p.106): the instrumented actions made the question accessible to the students; the teacher fostered their thinking processes by asking them the right questions at the right moment and the BMR pushed most of the students towards the understanding of this question. The use of software in this example has been complementary and not substitutive to that of paper and pencil environment. Using both has allowed to get an important goal. The dialectic between what the students have foreseen in the paper and pencil environment and what they are seeing within the TI-*n*spire environment poses the problem of the independent variable and gives fuel for solving it. The teachers pushes the best groups towards the solution like in L's group, while it is the final discussion orchestrated by the teacher to support almost all students to the right solution.

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