January 2009

- June 2011

CAYEN

Final Report

Matthias Zeller

Institute for Mathematics, University of Education

Freiburg, Germany; zeller.matthias@ph-freiburg.de

Bärbel Barzel

Institute for Mathematics, University of Education

Freiburg, Germany; barzel@ph-freiburg.de

Paul Drijvers

Freudenthal Institute for Science and Mathematics Education

University of Utrecht, Utrecht, Netherlands; pauld@fi.uu.nl

Table of Contents

Introduction 3

1 Theoretical background 4
1.1 Competences when learning algebra with CAS 4
1.1.1 Algebraic competences: conceptual knowledge, symbol sense, algebraic inside 4
1.1.2 Instrumental knowledge 5
1.2 Integration of technology in the learning process 6
1.2.1 Didactic issues to the use of technology 6
1.2.2 Technical issues of CAS and GC in elementary algebra 8

2 Methods and design 10
2.1 Design of the whole study 10
2.2 Overview of the single research cycles 10

3 Results 14
3.1 Influences of CAS on teaching and learning algebraic competences 14
3.2 The main influences of CAS on the learning of elementary algebra 18

4 Activities 25

5 References 27

Introduction

Since the first CAYEN-ideas have been developed till now more than two years have past. During two empirical research cycles and an extensive literature project the research field became much more focused and many aspects to the integration of CAS in grade seven came out. During the different phases the goal which was strived was the same all time: The results should give hints to the central question, if the use of CAS when learning elementary algebra is gainful. To answer this question two groups of pupils have been observed in a comparison design: one TI-nspire CAS group and one TI-nspire non-CAS group. The teaching unit which was inserted in both groups was to be as similar as possible and the data which came out were a strong basis to compare the groups in deeper analysis.

The qualitative results of research cycle one figured out many aspects which appeared in the pupils' work with CAS respectively graphing calculators (GC). Furthermore we had an empirical basis to decide which technical differences between the two TI-nspire versions have a strong influence on the pupils' learning. This makes it possible to reduce the whole research field a lot and to focus on the most important aspects. The outcome of this research cycle is that clear hints to the influence of CAS on the pupils learning can be observed and that in many mathematical activities of the pupils CAS helps to acquire algebraic competences. In addition, the pupils of our investigations developed instrumental knowledge, which is a complex mixture of mathematical and media skills. The development of a quantitative instrument to confirm our qualitative hypothesis was mastered in research cycle two. Theoretical derivations completed by empirical observations generated a multifaceted set of tasks which makes it possible to display algebra and CAS related competences. The broad theoretical background strengthens our own results and shows relations as well as discrepancies to other projects.

In chapter one of this report the most important parts of the theoretical background are presented. Chapter two deals with the design of the whole study and deepens the single research cycles. The outcome of all the work is presented in chapter three. Further activities and literature concerning CAYEN are displayed in the overview of chapter four.

1 Theoretical background

1.1 Competences when learning algebra with CAS

1.1.1 Algebraic competences: conceptual knowledge, symbol sense, algebraic inside

The field of algebraic competences is split into two parts: symbol sense and basic skills. Basic skills are necessary for manipulating expressions with variables, for solving equations and for using algebraic language. Beside these skills there are other important competences which are indispensible when dealing with variables, especially when solving problems or realistic tasks. These skills are described by the term ‘symbol sense’ (Arcavi 2005).

Algebraic competences	
Basic skills	Symbol sense
Syntactical skills allow for calculating with numbers and variables, e.g.: • solving equations • manipulating expressions in regard to certain rules • understanding and using algebraic notation and language	Non-syntactical skills allow for flexible handling of algebraic objects, e.g.: • the perspective on algebra and on its relation to arithmetic • the evaluation of algebra in comparison to other representations • cognitive activities and conceptual understanding in algebra

Table 1

The competences, which are displayed on the left-hand side of table 1, can be seen as procedural and declarative knowledge in algebra. Routine works, like solving equations or simplifying expressions, are mostly done with certain procedures, which can be used in different tasks. When using these procedures one does not think about their background and they can be known by heard. Also part of basic algebraic skills is declarative knowledge which refers to mathematical facts, like usual notations or terms. To understand where algebraic procedures and facts come from, in which way they belong to each other, when and how they can be used meaningful and which alternative ways exist, conceptual knowledge is necessary. The competence which is used therefore is symbols sense (left-hand side of table 1). Although Arcavi (2005, p. 43) points out that “the characterization of symbol sense is not fully developed” he proposes a possible definition with describing several competences which are surely included (Arcavi 2005, p. 42):

1. Friendliness with symbols
2. An ability to manipulate and also to ‘read through’ symbolic expressions as two complimentary aspects in solving algebraic problems
3. The awareness that one can successfully engineer symbolic relationships
4. The ability to select one possible symbolic representation for a problem
5. The realisation of the need to check for the symbol meanings during the implementation of a procedure
6. The realisation that symbols can play different roles in different contexts

According to table 1, the field of symbol sense is split into three parts: the perspective on algebra and on its relation to arithmetic, the evaluation of algebra in comparison to other representations and cognitive activities and conceptual understanding in algebra. On the one hand, some of Arcavi’s aspects are related to the perspectives that pupils have on algebra (e.g. aspect 1). On the other hand, he describes competences, which are required in order to understand the background of algebraic objects and which rely on the pupils’ conceptions of these objects (e.g. aspect 6). The other aspects can be seen as well under ‘perspective’ and under ‘conceptual understanding’, for example aspect 2. A part of aspect 2 is “to ‘read through’ symbolic expressions” (Arcavi 2005, p. 43). This is an important part of ‘conceptual understanding of algebraic objects’. In the same aspect is mentioned ‘manipulation’ and ‘reading through’ are “two complimentary aspects” (Arcavi 2005, p. 43). Being aware of having complimentary aspects of algebraic objects can be classified under the heading ‘perspectives on algebra’, because it influences the motivation and habit to use algebra. In his aspects is as well integrated the competence to switch between different equivalent expressions and to have an insight into structures of expressions with variables. Many authors see this as a central competence when dealing with CAS and call it ‘algebraic insight’ (Ball and Stacey 2005). Especially when comparing equivalent expressions, which are produced technology-free or by CAS, ‘algebraic insight’ is central.

1.1.2 Instrumental knowledge

Many authors of general media pedagogic point out the importance of knowledge about media, skills in handling media and critical assortment of media (Fuglestad 2005). These competences can be adapted to the use of CAS when learning elementary algebra. Pupils not only have to be able to master mathematical procedures, but also to master the use of technology. When solving an open task it is necessary to choose an adequate representation (like graphs, spreadsheets or to expressions) and an adequate medium (like paper and pencil or CAS) as well as to be able to carry out mathematical procedures with CAS or technology-free. When solving open tasks some steps are done with technology and some steps are done without technology and all steps influence each other. In the whole process of solution, a complex and individual mixture of different competences is used. These competences can not exactly be categorized in technological or mathematical competences. Artigue (2004) uses the term ‘instrumental knowledge’ to describe this composition of mathematical and technological knowledge. The competence includes much more than been familiar with the handling of media. It is necessary to choose adequate media and application and to evaluate their use for the required purpose.

1.2 Integration of technology in the learning process

1.2.1 Didactic issues to the use of technology

Waits (2000) points out that:

“Some Mathematics becomes more important because technology requires it.

Some Mathematics becomes less important because technology replaces it.

Some Mathematics becomes possible because technology allows it.”

According to this statement and the competences which are described in chapter 1.1 there are four different opinions concerning the competences pupils should acquire in mathematics lessons. Upon Such opinions often teachers’ beliefs base and convincing them of the use of CAS (for example in teacher training) means to convince them of opinion 4.

1. In mathematics lessons pupils should only acquire mathematical skills and thereby technology disturbs them.

2. In mathematics lessons pupils should only acquire mathematical skills and thereby technology supports them.

3. In mathematics lessons pupils should acquire mathematical skills and media skills. They still should acquire the same mathematical skills as in technology-free lessons.

4. In mathematics lessons pupils should acquire instrumental knowledge, which means that some mathematical problems can only be solved with technology.

The question whether technology disturbs or supports pupils when learning elementary algebra (opinion 1 vs. opinion 2) is discussed in many research projects (Barzel 2006, Weigand and Bichler 2009, Bruder and Ingelmann 2009, Schmidt 2009, Leng 2003, Ball and Stacey 2004, Neill 2009, Kieran and Drijvers 2006). In general the results show clearly that pupils strongly benefit of the use of technology although it carries risks in some points. General media pedagogic strongly recommends the integration of technology in the single subjects and media skills must be seen as a part of modern mathematics lessons (opinion 1 and 2 vs. opinion 3 and 4). The questions whether all mathematical competences have the same importance as in technology-free lessons (opinion 3 vs. opinion 4) is answered in general mathematics didactics. Routine procedures take much time to be learned in the lessons and often make it hard to underline the background and the interdependence of mathematical objects. If syntactical work is done automatically, one can focus on the acquisition of conceptual and instrumental knowledge. Not all syntactical work has to be done technology-free and conceptual knowledge is in the focus of the lessons. When solving open problems, many ways can be chosen and the competences cannot clearly be categorized in media competences and mathematical competences. In the process of solution the pupils use a mixture of technological and mathematical competences and acquire instrumental knowledge.

In mathematics lessons, in which pupils acquire instrumental knowledge (opinion 4), the pupils decide if at all and how CAS is used in the sense of a general purpose tool. In contrast to that is the use of CAS as a learning environment. For example, when a digital worksheet is given to pupils the use of technology, the goal and the methods to achieve it are predetermined. In this case technology is used in the sense of a learning environment (Barzel et al. 2005). A contrasting case is the use of technology as a general tool which the students can deploy at will in any step of their work (Barzel et al. 2005; Doerr and Zangor 2000) and thereby instrumental knowledge is an important competence to have or to acquire. Programs like TI-nspire are useful tools due to the fact that they work well with a broad range of mathematical problems and each person can use them in their own way (Fuglestad 2005). At the beginning of the solving process of a realistic problem CAS can also support the development of a mathematical model. In further steps, the calculations can be done automatically and eventually can validate their results (Pierce and Stacey 2002; Noss and Hoyles 1996). In any case the handling of the technology should be known beforehand or supported by a reference book. Altogether a general tool can be integrated into the individual solving process in accordance with individual ‘mental schemes’ (Drijvers and Trouche 2008), which are an important part of instrumental knowledge. How to integrate the technology into the learning process, however, is subject to the learner. This ‘instrumental genesis’ (Drijvers and Trouche 2008) describes the transformation of an ‘artifact’ into an ‘instrument’ for example during the process of learning algebra. Not only the integration of technology in single solution processes is of interest here but also the integration in a whole teaching unit.

Powerful commands like ‘solve’ and the underlying mathematics in the frame of solving equations are crucial content of several lessons. At which time should pupils use such a command? Already before they know the mathematical background or only afterwards? In an early phase of the learning process technology supports pupils in exploring mathematical content. First, it offers the possibility to focus on non-syntactical skills, e.g. describing problems in mathematical expressions. The use of technology may even motivate pupils to use new ways of solving tasks even if syntactical steps of the solution process are not done by themselves.

Second, working this way gives the students hints to the background of syntactical procedures (e.g. when solving equations stepwise in the calculator). As a consequence, the desire may arise to understand how the calculator works and subsequently to replace its work by manual activities. Thus the calculator can be seen as a black-box (Buchberger 1990; Heugl et al. 1996) focusing on user input and technological output rather than on intermediate stages. In later phases of the problem solving process when syntactical skills are developed by the pupils, the mathematical work of the calculator becomes more transparent. The work transferred to technology is now conducting single manipulations (e.g. manipulating only single parts of an expression), detecting mistakes and controlling results. Even if some of the syntactical skills get lost, the pupils are able to master much more problems by using instrumental knowledge, than they could with technology-free work.

1.2.2 Technical issues of CAS and GC in elementary algebra

A categorization of programs and tools for learning and teaching is not easy (Pallack 2007), because many programs offer functionalities in different mathematical fields. For example, most geometry programs include as well possibilities for drawing and manipulating graphs dynamically. Also a crucial point is the border between graphic calculators (GC) and computer algebra systems (CAS). In TI-nspire non-CAS some functions are available, which clearly belong to the field of algebra, like using variables till a certain degree or solving equations numerically. But there are three points which clearly characterize a CAS:

- Automatic symbolic transformations of expressions with variables
- Output includes variables and exact symbolic numeric expressions

- Symbolic expressions are strongly linked to graphic and numeric ones

Most authors agree to point one and two, the only question is, if dynamic graphing and spreadsheets are included when using the term ‘CAS’. To hint at this point sometimes the terms ‘computer algebra’ (CA; without ‘S’) or symbolic calculators are used for Programs, which do not offer dynamic graphing and spreadsheets. In elementary algebra and arithmetic especially point one (automatic symbolic transformations) leads to some commands which are only possible when using CAS and not possible with GC:

Functionality (command, example) in TI-nspire	non CAS	CAS
Defining a value of a variable or a function (define f1(x)=3x+5)	✓	✓
Solving equations with one variable numerically (nsolve (3*x*+5=7,*x*))	✓	✓
Solving equations with several undefined variables (solve (3*x*+5=7a,*x*))	-	✓
Manipulating algebraic expressions considering the distributive law (expand; factor)	-	✓
Substituting variables by numbers or algebraic objects (3*x* I *x*=5*a*)	-	✓
Manipulating expressions or equations e.g. for solving stepwise ((3*x*+5=4*x*+6)-3*x*)	-	✓
Checking if equations with variables are correct or incorrect (true or false)	-	✓
Calculating with undefined variables and showing results with variables (*x*+2*x*-4a-3a)	-	✓

Table 2

2 Methods and design

2.1 Design of the whole study

The main question of the CAYEN-project is if the integration of CAS in the learning process of elementary algebra is gainful. To answer this question it is adequate to compare GC-pupils and CAS-pupils because the influence of the possibility of linking graphic and numeric representations on the learning of algebra is the same in both groups. The only difference is in the possibility to use the commands presented in chapter 1.2.2 and the possibility to link symbolic expressions with several variables with graphic and numeric representations. In table 3 (next page) is displayed, how the design of the project arises in the technical differences.

2.2 Overview of the single research cycles

The CAYEN-project is split into three phases: A qualitative, explorative observation in spring 09, a quantitative and qualitative observation in summer 10 and a literature project in winter 10 / 11. In Table 4 the aims, the frame and the methods are shown as well as the research questions which are answered and the data and the material which have been used. The right cleavage comprehends a short evaluation of the outcome and the consequences for next research phases. The cleavage ‘data and material’ illustrates which material has been produced for analyzing and further research.

<table>
<tr><td></td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr>
<tr><td rowspan="2"></td><td rowspan="2">lists and spread-sheets</td><td rowspan="2">dynamic graphing</td><td colspan="2">link of representations</td><td rowspan="2">automatic symbolic transfor-mations</td><td rowspan="2">exact solution</td></tr>
<tr><td>graphs – spread-sheets</td><td>+ symbolic ex-pressions</td></tr>
<tr><td>TI-nspire non-CAS</td><td>✓</td><td>✓</td><td>✓</td><td>-</td><td>-</td><td>-</td></tr>
<tr><td>TI-nspire CAS</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td></tr>
<tr><td colspan="7"></td></tr>
<tr><td colspan="7">Research cycle one (2009): observing all aspects
Observation and comparison of a non-CAS and a CAS-group
Finding out the main influences of CAS in elementary algebra</td></tr>
<tr><td colspan="7">Research cycle two (2010): observing aspect 4,5,6
Observation and comparison of a non-CAS and a CAS-group
Verifying the results of research cycle one by concentrating on the difference between CAS and GC</td></tr>
<tr><td colspan="7">Follow up investigation (2011): observing aspect 5
Observation and comparison of a CAS- and a technology-free-group
Finding out, if the results differ from the results of the Comparison with the GC-group</td></tr>
</table>

Table 3

Phase of CAYEN	aims / frame / methods	questions which are answered	data and material	evaluation of the outcome
research cycle one: Observation in spring 09	• Developing the teaching material and the test; finding hypothesis and questions; finding answers to these questions • explorative, qualitative comparison of two CAS-classes and one GC-class during 8 weeks • video supported open observation; grounded-theory analysis	• How do both technologies influence the pupils' perspective on algebra and their perception of its relation to arithmetic? • How do both technologies influence the pupils' evaluation of algebra compared to other representations? • How does the availability of CAS influence the pupils' cognitive activities and conceptual understanding in the field of algebra?	• teaching unit in the first version • test in the first version • pupils' written solutions of tests and some tasks of the teaching unit • Videos of the whole teaching unit	The research field became much more focused and the three topics in the research questions seem to be interesting points. The qualitative proofs are a useful beginning, but base only on single cases. In the data are good hints for the development of the teaching material. The test did not work well in the first version.

Research cycle two: Observation in spring 10	• developing the test and the teaching material; finding more proofs to the results of the first observation • qualitative observation of six CAS-pupils and six GC-pupils when solving one special task; pre- and post-test of 30 CAS-pupils and 30 GC-pupils • video observation during 90 minutes; data is not analyzed yet	• Same questions as in the first observation • Which written answers in tests are helpful to answer the research questions?	• teaching unit in the second version • test in the second version • some additional videos of single groups of pupils when solving tasks • written solutions of the tests and some tasks of the teaching material	The test works well. The videos are in good technical quality, but it is not necessary to collect them during the lessons. The differences of both groups are not as big as expected, what could be related to the fact that the pupils did use the calculator for only six weeks.
Literature project in winter 10	• Getting an updated overview of literature • Analyzing about 150 articles and books • Summarization and categorization of the texts	• Which studies can be used as a basis for CAYEN? • Do our results match, extend or reflect other results? • Do we get hints to the development of the teaching unit?	• data file with summarizations of some selected texts • data file with citations of the whole texts	The categories in our report can be used well for developing the teaching material and for the communication with the teachers. The relation between CAYEN and other projects became much clearer.

Table 4

3 Results

3.1 Influences of CAS on teaching and learning algebraic competences

In the following chapter there is an overview of the broad field of the influences of CAS on teaching learning and assessing. On the one hand some aspects base upon the results of research cycle one, on the other hand some aspects can be seen as the outcome of the literature project. The following claims are the product of analysis and categorizations of results published in the last ten years. Some of these claims are deepened by our own research. Claim 1 to claim 4 deal with the influences of CAS on learning algebra and illustrate that the competences, which pupils acquire in a CAS-supported learning environment, change and develop. The influences of CAS on teaching elementary algebra are discussed in claim 5 to claim 7 and it is shown that mathematics education with CAS strongly fulfills the allowances of current mathematics didactic.

1 CAS supports the development of conceptual knowledge

Conceptual knowledge is the background of symbol sense and algebraic inside and these competences are stimulated and required when working with CAS. When working with CAS pupils concentrate directly on the interdependence of the input and the output and are not disturbed by mistakes in intermediate steps (Zeller and Barzel 2010). Furthermore, the time between the input and the result is very short, what helps them to keep an overview of the whole task in mind. The goal is to master realistic tasks and problems and not to do the same algebraic routines several times (Zeller and Barzel 2010). If the output differs from the expectation of the pupils, a cognitive conflict arises and pupils can manage this by checking their results by technology-free work. If the same commands are used in different tasks it easy for pupils to recognize the relation between different mathematic fields. Not only the same commands, but also the same mathematic objects appear in CAS very multifaceted. For example, it is easily possible to replace variables by numbers, to use them for modeling and to work with them in symbolic transformations. In addition, it is sometimes necessary to replace them by other symbolic expressions, what is done rarely in technology-free work, but underlines the insertion aspect of variables. In the output the variables are presented in a common way and pupils do not doubt their appearance. Altogether learning with CAS helps pupils to develop meaningful concepts of algebraic objects, like variables, expressions or equations, which are an important part of conceptual knowledge and symbol sense.

(related literature: Kieran und Drijvers 2006, Cuoco und Levasseur 2003, Abdullah 2007, Zeller und Barzel 2010)

2 Technology-free competences may still be acquired in a CAS-supported education

Although these competences are not in the main focus of CAYEN, the relations to other research in this point are important for designing the teaching-material and for defining the boundary to our research field. Instrumental knowledge includes the ability to detect previously which media is useful to solve a task and for some tasks or parts of it paper and pencil are suitable. For example easy calculations, sketches of graphs or writing down whole solutions, is sometimes done more quickly in a technology-free way. These competences may be exercised in special technology-free parts of the lessons. Important is that not only procedural skills are practiced, but also conceptual work. For example algebraic transformations can be done in mind and questions like 'what happens if …' can answered in mind, too. Such phases may even refer to CAS. Questions like what 'what would you use it for …' or 'what would happen if a certain command is used' can activate pupils' reflections of CAS-work and help them to develop technology-free competences.

(related literature: Kieran und Yerushalmy 2004, Ingelmann 2009, Artigue 2004, Lagrange 2003)

3. The use of mathematical language is activated by CAS

Our own research as well as some literature report that pupils use CAS the terms of CAS-commands in their work without technology. In our studies we have seen, that pupils even write down which command they would have used, if they for example solve an equation. Furthermore this helps them to structure their work and to express explicitly what they are doing in the next lines of their written document. The notation of CAS differs to the traditional notation. In respect to the technical handling this is sometimes necessary, but it does not at all constrict the mathematical core. For the future it would even be possible that both notations develop in the same direction and will not differ anymore. Written communication about mathematics is in CAS-work on a high level. Because to handle CAS it is necessary to understand the background of the algebraic language and to express what shall be done in a mathematical way by typing in the right commands. Thereby, for example the aspect the an equation may be solved for different variables becomes clearly when typing in ',x' after the equation.

(related literature: Drijvers 2003, Greefrath 2007, Ball und Stacey 2005, Zeller und Barzel 2010)

4. Technical skills complete mathematical competences meaningfully in the sense of instrumental knowledge

The process of developing instrumental knowledge is described in the model of instrumental genesis. When solving a certain task one takes an artifact and uses it as a certain instrument. Thereby one individually uses and develops mental schemes, which are part of instrumental knowledge. This process only happens, if a task is given, what means that instrumental knowledge in mathematics mainly is acquired when solving mathematical tasks with adequate instruments. Reflecting the use of instruments and mathematical procedures is in a CAS-supported education an important part. The reflection enriches lessons with additional topics like discussions about the most effective ways (including media and mathematical procedures) for solving tasks.

(related literature: Fuglestad 2005, Ball und Stacey 2005, Zbiek 2003, Drijvers und Trouche 2008)

5. CAS advantages a global and genetic composition of mathematical topics

Traditionally mathematical topics are split into small pieces and these pieces are taught little by little. In contrast to that is the genetic teaching approach, in which one begins with tasks for which the pupils do not know the way how to solve it. This way and the necessary mathematical procedures have to be developed by the pupils their selves. A big advantage of this approach is that the pupils learn self-dependent and do not only repeat what the teacher told. A problem is that often many mathematical topics are discovered by the pupils, but cannot be deepened in the further lessons. In our own studies we experienced that pupils were able to solve mathematic problems on high level with CAS. Interestingly some of these tasks were taken out of textbooks for higher level students, but the CAS-groups already could solve them. Sure they needed some help of CAS for mastering the algorithmic procedures (e.g. solving equations of higher degree), but they did the main challenges of the tasks on their own. For example in modeling tasks, they described realistic contexts in useful algebraic models, although they could not solve these models on their own. Firstly, this involves the big mathematical ideas already at the beginning of the learning process and not only after acquiring procedural knowledge. Secondly, the global relation of mathematical procedures is underlined because they can be used parallel, although they cannot be solved in a technology-free way. In Algebra therewith symbol sense is being developed already at the beginning and during the whole learning process.

(related literature: Goldenberg 2003, Buchberger 1990, Kendal and Stacey 2001, Zeller and Barzel 2010)

6. The integration of open tasks and the use of individual ways of solving them is supported by CAS

Open tasks have to be solved in individual ways and in self-dependent work. A Problem is that it is difficult for teachers to check the individual solutions. In the CAYEN-studies we have seen that the pupils check their solutions on their own with CAS and that they indeed recognize mistakes and are able to correct them. From a theoretical point of view one could say that the amount of ways of solving tasks with CAS is two times higher as in technology-free work, because each mathematical approach can be done with technology or without. Empirical studies show, that the amount of different solutions is even higher, because most pupils use an individual mixture of technology-supported and technology-free work in the single steps of the process of solution. CAS can be seen as a mathematical laboratory in which pupils can do experiments like trying out different inputs and different commands. This kind of work can be compared to the kind of work of natural science and supports a dynamic concept of the whole mathematics.

(related literature: Challis and Gretton 2002, Doerr and Zangor 2000, Böhm et al. 2004, Heid and Blume 2008, Drijvers 2003, Laakman 2008, Leng 2003, Zeller and Barzel 2010)

7. CAS puts a new complexion on teaching methods

According to the genetic teaching approach and to the open tasks, the teaching methods have to be adapted as well. A problem, but also a chance is that the current generation of the teachers did not learn mathematics with technology. On the one hand, when beginning to insert CAS one cannot teach in the usual way and especially not in the way teachers were taught their selves. But on the other hand, teachers are forced to rethink their teaching. There is the chance to change old habits according to the allowances of mathematics didactics and CAS can be seen as a catalyst therefore. In detail this means that self-dependent phases of the lessons are enriched by informal talks between the pupils about technology and frontal phases are enriched by discussions about technical issues and instrumental topics. To encourage teachers to begin to use technology it is necessary to point out that it really is more preparation of material for lessons at the beginning, but in the lessons the teacher is discharged by CAS and has the possibility to give attention to the pupils' problems. All the effort is worthwhile, because the lessons are more flexible, more self-dependent and therewith increase much.

(related literature: Aldon 2010, Ball and Stacey 2005, Heid and Blume 2008, Neill 2009, Barzel 2006)

3.2 The main influences of CAS on the learning of elementary algebra

In research cycle one we observed that the pupils' activities in both groups differed mainly in the black-box-phases of the teaching unit. In these phases the pupils had the possibility to use unknown commands to solve the algebraic models they had constructed on their owns. When working with GC, this approach is also possible, when learning to work with graphs or spreadsheets, but not when learning to use algebraic procedures. The pupils of the GC-group did not use algebraic models that much as the CAS-pupils, because they could not solve their models. The consequence was that the CAS-pupils were encouraged to use new mathematical approaches, especially symbolic ones, while the GC-pupils sometimes even replaced algebraic work by graphic or numeric approaches, which they already were familiar with. The black-box-approach supported the pupils in two aspects: First, when learning in a genetic way CAS-pupils mastered new challenges in algebra well and had not many problems in self-dependent phases. Second, it was possible to get an overview of a topic at the beginning, by trying out new commands. This aspect means that with CAS it is easily possible to learn many aspects of a mathematical topic parallel. The single aspects are deepened in later phases of the learning process. The CAS-pupils started already at the beginning of a new topic to think about unknown commands and to ask about what they can be used for, although this was not activated by the teachers. A challenge of the observation was, that on the one hand the teaching material had to be as similar as possible, to be able to compare the both groups. On the other some tasks could only be inserted in the CAS-group. As mentioned in chapter 2.1, we are planning a follow up study. In this study we will not respect for the comparison much. This gives us more freedom in the design of the teaching material and CAS-specific tasks can be integrated stronger in the teaching unit.

At the beginning of CAYEN we had the hypothesis that a strong influence of CAS is on the switch and link of representations. In the qualitative observations of research cycle one this could not be confirmed well, but another phenomenon appeared. The pupils evaluated algebra in another way as their GC-school mates. In our analysis of pupils' written comparisons of graphic, numeric and symbolic representations, we noticed much more advantages of algebra in the CAS-group. An effect was that they were more motivated to use algebra and inserted it more often in open tasks.

The results of research cycle one are categorized in three claims and according to these claims we developed the tasks of the test in research cycle two. Below the results of research cycle one are presented shortly and the related tasks for the test are shown.

1. CAS and GC and the perspective on algebra

When learning elementary algebra the pupils do a step from arithmetic to algebra and ideally use their arithmetic competences to develop algebraic ones. CAS-pupils master this step more easily. They accept the output of the calculator as a common means of expression and realize the relevance of algebra. Furthermore they perceive already at the beginning the versatility of algebraic work in contrast to arithmetic approaches. By discovering the output of the calculator they get to know many commands and algebraic transformations and it did not matter that they could do them all in a technology-free way. GC-pupils sometimes even had difficulties in accepting that the same underlying rules are valid in algebra and arithmetic. They argued that their calculators should be able to handle expressions with variables, if the same rules would be valid.

To measure the pupils' step from arithmetic to algebra, it is necessary to measure on the one hand their algebraic skills, but also their arithmetic skills. The algebraic skills are for example measured in the tasks of claim 3; the arithmetic skills of the pupils are measured for example in the task below. This task focuses on the pupils' perceptions of arithmetic structures and on the ability to recognize and use mathematical objects in different calculations. The given results can be used in the calculations. In task 13)f) for example the pupils have to use the given calculation on the right hand side, to restructure the arithmetic equation and to recognize that the result easily can be seen and is 2438. The numbers are that high, because the pupils really shall use the structure and not calculate intermediate results in mind or in written work.

13) *Lena had to do three complicated calculations in their homework. Luckly she had an calculator. These are the tasks and Lena's correct results:*

356 + 895 + 3965 + 10000 = 15216 *756 – 210 + 84 = 630* *2438 + 517 – 767 – 80 = 2108*

Do the following calculations without your calculator. Lena's calculations can help you.

a) *3965 + 10356 + 895 =*

b) *(762 + 78 – 210) · 100 =*

c) *895 + 356 + 3965 =*

d) *736 – 190 + 84 =*

e) *2438 + 517 – 847 =*

f) *2108 – 517 + 767 +80 =*

2. CAS and GC and the evaluation of algebraic representation

In our analysis of the pupils’ written documents in research cycle one, we wondered about some solutions of the GC-group. In a group work of an open real-life problem, some pupils started to work algebraically, but changed to graphic approaches in further steps. In a deeper analysis of the videos, we recognized that this change of representations had no mathematical background. The GC-pupils only wanted to avoid syntactical work. This syntactical work is done automatically in the graphs and geometry application of TI-nspire when working with graphic representations but cannot be done automatically when working with an algebraic approach with TI-nspire non-CAS. In further lessons of the teaching unit, the teachers started a classroom discussion about the advantages and disadvantages of the numeric, the graphic and the algebraic approach. The pupils of the GC-group saw more disadvantages of the algebraic approach. Furthermore most of their arguments had no mathematical background, like the overview of the task or the effectiveness. They reported that algebra is more work and that it is hard to verify the solutions. In the written test of research cycle two we got a further hint for this phenomenon. In task 14) (below) the pupils had to compare different solutions of a task. Our analysis of the pupils’ documents showed again that the argumentation concerning algebra of the CAS-pupils included more mathematical arguments and was more objective as the argumentation of the GC-pupils.

The adjectives, terms and questions which are presented in task 14)a)-f)are related to the pupils’ answers in the classroom discussion of research cycle one. Therewith we could be sure, that the pupils understand the questions. Although we did not analyze deeply the outcome concerning the graphic and the numeric approach, we decided to integrate a comparison of the representations and not only an evaluation of algebra as a single representation. In our test of the pilot phase we recognized that the pupils mastered easily this comparison, but did not understand what to do at a task concerning the evaluation only of algebra.

14) *The class 7b should compare the fees for mobile phones (table below).*

The pupils‘ solutions of this task are presented below. Three groups took three different ways, however they had all the same, correct solution: phonig for 18 minutes per month costs at both offers about 15€. 21fo ne phones more, ‚phonic‘ is cheaper than ‚mobile‘.

	Tarif „mobile“	Tarif „phonic“
Grundgebühr pro Monat	3, 90 €	5,90 €
Preis pro Minute	0,60 €	0,49 €

Look at first at all the three solutions on the following pages. How much do you like them? Answer then the following questions (a-g). Write your answers on the lines beside the solutions.

a) Which part of the solution do you understand, which part do you not understand?

b) How exactly is the solution?

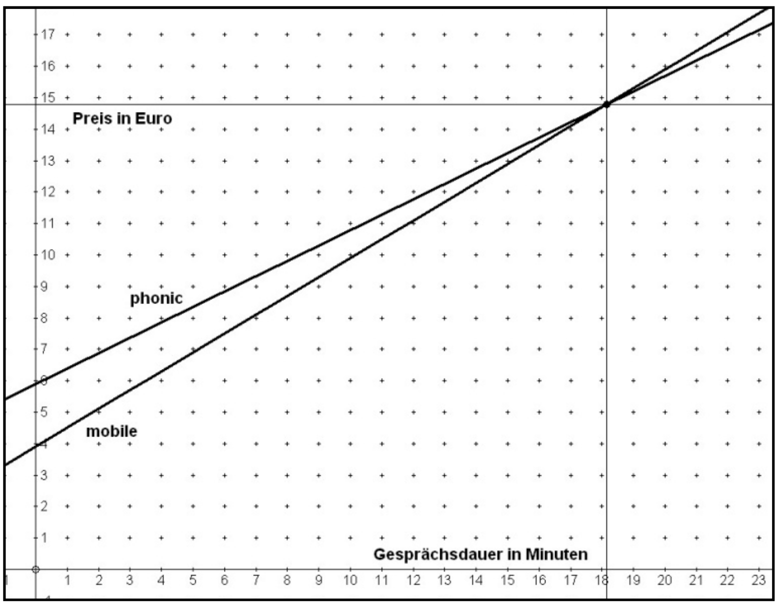
c) How fast can one see the result?

d) How good is the overview over the whole problem?

e) How ‘elegant’ is the solution?

f) How long does it take to do the solution by hand? And by calculator?

g) How difficult is it to do the solution by hand? And by calculator?



Minutes	mobile	phonic
1	4,50 €	6,39 €
5	6,90 €	8,35 €
10	9,90 €	10,80 €
15	12,90 €	13,25 €
16	13,50 €	13,74 €
17	14,10 €	14,23 €
18	**14,70 €**	**14, 72 €**
19	**15,30 €**	**15,21 €**
20	15,90 €	15,70 €
25	18,90 €	18,15 €

Time of phone call in minutes: x

Fees mobile = 3,90 + 0,60 · x

Fees phonic = 5,90 + 0,49 · x

When are both fees the same?:

$3,90 + 0,60 \cdot x = 5,90 + 0,49 \cdot x \quad | \; - 3,90$

$0,60 \cdot x = 2,00 + 0,49 \cdot x \quad | \; - 0,49 \cdot x$

$0,11 \cdot x = 2,00 \quad | \; : 0,11$

$\underline{x = 18,18}$

Fees at 18,18 min.: <u>14,81 €</u>

3. CAS and cognitive activities in the field of algebra

Especially in the video analysis of research cycle one, we observed that the thoughts of pupils when using CAS are on a high algebraic level and include a lot of conceptual aspects. For example when solving an equation stepwise by CAS, at first pupils think and talk about how the next line should look like and do not type in a transformation without reflecting. In a second step they think about how this expression can be produced and then they type in how to manipulate the equation on both sides. This kind of work is not possible if the pupils only learn rules and syntactical procedures. The competences are to recognize structures of expressions, to focus on single variables or parts of them and to imagine expressions which shall be produced. These activities stimulate the acquirement of conceptual knowledge and of a sense for structures of expressions. When solving equations in a technology-free way, pupils used the commands of CAS in their exercise book. Not only have the commands of the calculator been adopted in the exercise books, but also the kind of work. We observed in the videos that at the beginning the work of the pupils was not conducted by algorithms and that they started to discover and to develop individual ways of solving equations and algebraic manipulations.

Some CAS-pupils produced very abstract algebraic solutions of open problems. In intermediate steps, they inserted complicated expressions for variables and used their expressions and equations for finding solutions. Sometimes they replaced variables with other letters, to hint at their meaning in the context of the problem. Therefore they used the 'within'-command (I), what hints at the insertion aspect of variables. It is interesting that they did not forget what they used the variables for during the process of solution and had no problems in answering the overall questions with their CAS-results.

To measure if the pupils focus on the structure of expressions or if they work with algorithmic procedures we developed task 12 for the test. In this task the solution of an equation is presented and the pupils have to decide which equations have the same solution. Some pupils solved the new equation and others used the presented result and compared the structures.

12) *The solution of the equation* $11x - 17 = 19 + 7x$ *is* $x = 9$.

a) *Of which of these equations is the solution 9 , too? Mark with a cross.*

O $21p - 17 = 19 + 17p$ O $4y - 17 = 19$

O $11g + 17 = 19 - 7g$ O $19 + 7v = 11v - 17$

O $11f + 19 = 7f - 17$ O $a + 11k - 17 = a + 19 + 7k$

b) *How did you solve this task?*

Beside such inner mathematical tasks, we also tried to measure the pupils' competences in understanding models of realistic situations. Task 9 presents a model, in which the variable p is used to describe the price and the number of persons. The solution of this equation is how many guests have to come, that the price in € equals the number of persons, what means that every persons costs 1 €. In the task it only was asked for the model and the pupils have to recognize that it is not possible that the solution is a single number. The pupils have to comprehend the model 9)a), to evaluate the result 9)b) and propose a right algebraic model (9c).

***9)** Jana wants to rent the whole cream parlor on her birthday and to pay one scoop of ice cream for every guest. Tehrefore she got the following offer: 30 € rent for the room and 0.80 € per person for the ice cream. She wonders: „whart is the PRICE, if how many PERSONS will come?“ With her calculator she solves the following euqation:*

$p = 30 + p \cdot 0.80$

$p = 150$

Jana is surprised about the result and cannot remember what exactly she wanted to calculate.

***a)** Describe the single parts of Jana's equation. What does the p express?*

$$P \quad = \quad 30 \quad + \quad p \cdot 0.80$$

***b)** What means Jana's result concerning the persons and the price?*

***c)** Which equation would you have developed for Jana's problem?*

Task 12 and 9 are much closed, what on the hand gives the possibility to measure very exactly and many pupils, but what on the other hand avoids that some of the pupils individual thoughts concerning algebra can be seen. During the development of the test we recognized that most pupils have problems in answering very general and open questions on their perspective on algebra. But it is possible to give them a concrete situation or task and to let them reflect in a general way. In task 15 a concrete calculation is presented and the pupils have to answer the general question about the differences and similarities between arithmetic and algebra. In their answers they can refer to the task but also give hints to their general perspective on algebra and to the relation to arithmetic.

***15)** When calculating with numbers it is possible to use tricks.*

In this example one factored out.

$3 \cdot 13 + 3 \cdot 7$
$= 3 \cdot (13 + 7) = 3 \cdot 20 = 60$

Is it also possible to use such tricks when calculating with variables (what means with letters)?

Is there a difference between calculating with variables and calculating with numbers?

4 Activities

ICTMT 9 presentation	July 6th – 9th 09 Metz, France	At the “Ninth International Conference for Technology in Mathematics Teaching” the whole project was presented by B. Barzel and M. Zeller as well as discussed with researchers and teachers. The presentation is attached to this report.
ICTMT 9 paper	May 09	The project and some first impressions are introduced in the paper of the conference (available on www.ictmt9.org and in the attachment of this report).
Article in TI-Nachrichten (German)	Autumn 09	The article is already finished and includes an introduction of the project and aspects of the insertion of TI-nspire in teaching material and lessons (see attachment).
Mathe für alle	October 15th 09 Freiburg, Germany	Conference for teachers and students at the university of education, Freiburg. The use of TI-nspire and TI-nspire CAS will be introduced and the teaching unit will be discussed in the frame of a workshop.
Sharing Inspirations	November 20th, 21th 09 Frankfurt, Germany	Conference organized by T^3 for teachers which work with TI-nspire technology. The project will be introduced and the teaching material will be discussed in the frame of a workshop.
GDM 2010 presentation	March 08th, 12th 10 Munich, Germany	On the conference the results of the research cycle one have been presented and the ongoing of the project was discussed.

GDM 2010 paper	April 2010	In the proceedings of the GDM-conference the state of the art of CAYEN was presented.
retraining of trade school teachers	March 10th 10 Stuttgart, Germany	In workshop for trade school teachers in Baden-Württemberg the possibilities of TI-nspire CAS and the outcome of research cycle one have presented.
GDM summer-school	September 29th – Oktober 01th 10 Soest, Germany	The project was presented to other PHD-students and supervisors. The results of research cycle one and the ongoing of CAYEN was discussed.
Article in ZDM	October 2010	Article in the international journal "Zentralblatt der Didaktik der Mathematik". The whole project is introduced and the results of research cycle one are presented.
Presentation for the ministry of Thuringia	January 20th 2011 Erfurt, Germany	The outcome of the literature project was presented to the ministry of Thuringia. Some of the CAYEN-results were used to illustrate the basis of empirical research.
GDM 2011 presentation	February 21th-25th Freiburg, Germany	The literature project and the relations to CAYEN have been presented and discussed with other German researchers.
GDM 2011 paper	March 2011	The results of the literature project are discussed. Single aspects of CAYEN are integrated in the claims.
Publication of the literature work	Summer 2011	The influences of CAS on teaching, learning and assessing are presented. Requirements for a successful integration of CAS are described as well as consequences for the institutional frame.

5 References

Abdullah, L. (2007): Procedural Knowledge in the Presence of a Computer Algebra System (CAS): Rating the Drawbacks Using a Multi-factorial Evaluation Approach. In: *The International Journal on Mathematics Education (ZDM Mathematics Education)* 14 (1), S. 14–20.

Aldon, G. (2010): Handheld calculators between instument an document. In: *The International Journal on Mathematics Education (ZDM Mathematics Education)* 42 (7), S. 733–745.

Arcavi, A. (2005): Developing and Using Symbol Sense in Mathematics. *For the Learning of Mathematics, 25*(2), 50-55.

Artigue, M. (2004): The integration of computer technologies in secondary mathematics education. In: Wang Jianpan et al. (Hg.): Trends and challenges in mathematics education. Shanghai: East China Normal University Press, S. 209–222.

Ball, L.; Stacey, K. (2004): A new practice evolving in learning mathematics: Differences in students' written records with CAS. In: Marit et al Johnsen Høines (Hg.): Proceedings of the 28th International Conference of the International Group for the Psychology of Mathematics Education. PME 28, Bergen, Norway, July 14□18, 2004, Bd. 2. Bergen: Bergen University College, S. 87–94.

Ball, L.; Stacey, K. (2005): Teaching strategies for developing judicious technology use. In: William J. et al Masalski (Hg.): Technology-supported mathematics learning environments. Sixty-seventh yearbook. Reston, VA: National Council of Teachers of Mathematics, S. 3–15.

Barzel, B. (2006): Mathematikunterricht zwischen Konstruktion und Instruktion: Evaluation einer Lernwerkstatt im 11. Jahrgang mit integriertem Rechnereinsatz. In: *Journal für Mathematik-Didaktik (JMD)* 27 (3-4), S. 321–322.

Barzel, B., Hußmann, S., Leuders, T. (2005): *Computer, Internet & Co im Mathematikunterricht*. Berlin: Cornelsen Scriptor.

Buchberger, B. (1990): Should students learn integration rules? *Sigsam Bulletin, 24*(1), 10-17.

Böhm, J.; Forbes, I.; Herweyers, G.; Hugelshofer, R.; Schomacker, G. (2004): The case for CAS. Münster: Westfälische Wilhelms-Universität Münster.

Bruder, R.; Ingelmann, M. (2009): CAliMERO aus Sicht der Forschenden. In: *Mathematikunterricht (Seelze)* 55 (4), S. 13–19.

Challis, N.; Gretton, H. (2002): Expressive and explicit CAS - is it enough? In: *The Internation Journal of Computer Algebra in Mathematics Education* 9 (2), S. 155–166.

Cuoco, A.; Levasseur, K. (2003): Classical Mathematics in the Age of CAS. In: James T. Fey, Al Cuoco, Carolyn Kieran, Lin McMullin und Rose Mary Zbiek (Hg.): Computer algebra systems in secondary school mathematics education. Reston, VA: National Council of Teachers of Mathematics, S. 97–116.

Doerr, H.; Pratt, D. (2008): The Learning of Mathematics and Mathematical Modeling. In: M. Kathleen Heid und Glendon W. Blume (Hg.): Research on Technology and the Teaching and Learning of Mathematics: Syntheses, Cases, and Perspectives. Research Syntheses. 2 Bände. Charlotte, North Carolina: National Council of Teachers of Mathematics (NCTM) (1), S. 259–285.

Drijvers, P. (2003): Algebra on Screen, on Paper, and in the Mind. In: James T. Fey, Al Cuoco, Carolyn Kieran, Lin McMullin und Rose Mary Zbiek (Hg.): Computer algebra systems in secondary school mathematics education. Reston, VA: National Council of Teachers of Mathematics, S. 241–268.

Drijvers, P.; Trouche, L. (2008): From artifacts to instruments: A theoretical framework 66 behind the orchestra metaphor. In: M. Kathleen Heid und Glendon W. Blume (Hg.): Research on Technology and the Teaching and Learning of Mathematics: Syntheses, Cases, and Perspectives. Research Syntheses, Bd. 1. 2 Bände. Charlotte, North Carolina: National Council of Teachers of Mathematics (NCTM) (1), S. 363–391.

Fuglestad, A. (2005): Students' choice of tasks and tools in an ICT rich environment. In: Marianna Bosch (Hg.): Proceedings CERME4 conference february 2005.

Goldenberg, E. (2003): Algebra and Computer Algebra. In: James T. Fey, Al Cuoco, Carolyn Kieran, Lin McMullin und Rose Mary Zbiek (Hg.): Computer algebra systems in secondary school mathematics education. Reston, VA: National Council of Teachers of Mathematics, S. 9–32.

Greefrath, G. (2007): Computeralgebrasysteme und Prüfungen. In: Beiträge zum Mathematikunterricht. Vorträge auf der 41. GDM Tagung für Didaktik der Mathematik. Hildesheim: Franzbecker, S. 55–58.

Heid, K.; Blume, G. (Ed.) (2008): Research on Technology and the Teaching and Learning of Mathematics: Syntheses, Cases, and Perspectives. Research Syntheses. 2 Bände. Charlotte, North Carolina: National Council of Teachers of Mathematics (NCTM) (1).

Heugl, H., Klinger, W., Lechner, J. (1996): *Mathematikunterricht mit Computeralgebra Systemen.* München: Addison-Wesley (Deutschalnd).

Ingelmann, M. (2009): Evaluation eines Unterrichtskonzeptes für einen CAS-gestützten Mathematikunterricht in der Sekundarstufe I. Dissertation. Berlin: Logos Verlag.

Kendal, M.; Stacey, K. (2001): The Impact of Teacher Privileging on Learning Differentiation with Technology. In: *International Journal of Computers for Mathematical Learning* 6 (2), S. 143–165.

Kieran, C.; Drijvers, P. (2006): The co-emergence of machine techniques, paper-and pencil techniques, and theoretical reflection: a study of cas use in secondary School algebra. In: *International Journal of Computers for Mathematical Learning* 11 (2), S. 205–263.

Kieran, C.; Yerushalmy, M. (2004): Research on the role of technological environments in algebra learning and teaching. In: Kaye Stacey, Helen Chick und Margaret Kendal (Hg.): The future of the teaching and learning of algebra. The 12th ICMI study. Boston: Kluwer Acad. Publ. (New ICMI studies series, 8), S. 99–152.

Laakmann, H. (2008): Multirepräsentationsprogramme im Mathematikunterricht. In: *Mathematikunterricht (Seelze)* 54 (6), S. 44–49.

Lagrange, J.-B. (2003): Learning Techniques and Concepts Using CAS: A Practical and Theoretical Reflection. In: James T. Fey, Al Cuoco, Carolyn Kieran, Lin McMullin und Rose Mary Zbiek (Hg.): Computer algebra systems in secondary school mathematics education. Reston, VA: National Council of Teachers of Mathematics, S. 269–285.

Leng, N. (2003): Effects of Computer Algebra System on Secondary Students' Achievement in Mathematics: A Pilot Study in Singapore. In: *The Internation Journal of Computer Algebra in Mathematics Education* 10 (4), S. 235–250.

Neill, A. (2009): Key Findings From the CAS Pilot Programme. In: *The New Zealand Mathematics Magazine* 46 (1), S. 14–27.

Noss, R.; Hoyles, C. (1992): Afterword: Looking back and looking forward. In: Richard Noss und Celia Hoyles (Hg.): Learning mathematics and Logo. Cambridge, MA: MIT Press, S. 427–468.

Pallack, A. (2007): Die gute CAS-Aufgabe für die Prüfung. In: Beiträge zum Mathematikunterricht. Vorträge auf der 41. GDM Tagung für Didaktik der Mathematik. Hildesheim: Franzbecker, S. 90–93.

Pierce, R.; Stacey, K (2002). The algebra needed to use computer algebra systems. *The Mathematics Teacher, 95*, 622-627.

Schmidt, K. (2009): Mathematics Education with a Handheld CAS - The Students' Perspective. In: *International Journal for Technology in Mathematics Education* 17 (2), S. 105–110.

Waits, B. (2000): Instead of an introduction. ACDCA Proceedings Portoroz.

Weigand, H.-G.; Bichler, E. (2009): Der Modellversuch Medienintegration im Mathematikunterricht (M") an bayerischen Gymnasien. 100. MNU Kongress Regensburg. Online verfügbar unter http://www.didaktik.mathematik.uniwuerzburg. de/weigand/M3_Projekt_Materialien/MNU-Regensburg-2009-Weigand%20Bichler.pdf, zuletzt geprüft am 18.11.2010.

Zbiek, R. (2003): Using technology to foster mathematical meaning through problem solving. In: Harold L. Schoen (Hg.): Teaching mathematics through problem solving. Grades 6-12. Reston, VA: National Council of Teachers of Mathematics (NCTM), S. 93–104.

Zeller, M.; Barzel, B. (2010): Influences of CAS and GC in early algebra. In: *The 71 International Journal on Mathematics Education (ZDM Mathematics Education)* 42 (7), S. 775–788.